JOURNAL OF APPROXIMATION THEORY 35, 311–354 (1982)

Successive Approximation and Linear Stability Involving Convergent Sequences of Optimization Problems

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Communicated by Carl de Boor

Received May 22, 1981

A certain convergence notion for extended real-valued functions, which has been studied by a number of authors in various applied contexts since the latter 1960s, is examined here in relation to abstract optimization problems in normed linear spaces. The main facts concerning behavior of the optimal values, the optimal solution sets and the ε-optimal solution sets corresponding to "convergent" sequences of such problems are developed. General linear perturbations are incorporated explicitly into the problems of the sequence, lending a stability-theoretic character to the results. Most of the results apply to nonconvex minimization.

1. INTRODUCTION

This paper treats "convergent" sequences of implicitly constrained, linearly perturbed optimization problems of the form

$$P(v)\text{: minimize } f(x) - \langle x, v\rangle \text{ over } x \in X. \tag{1.1}$$

We assume $f: X \to [-\infty, +\infty]$ and that X is a real normed linear space with topological dual V. We write $\langle x, v\rangle$ to denote the value of $v \in V$ at $x \in X$. Our objective is to examine the behavior of the *optimal value* of $P(v)$,

$$\omega(v) = \inf_x \{f(x) - \langle x, v\rangle\}, \tag{1.2}$$

and for $\varepsilon \in [0, +\infty)$ the sets of ε-*optimal solutions* of $P(v)$,

$$\Omega(v, \varepsilon) = \{x \in X \mid f(x) - \langle x, v\rangle \leqslant \varepsilon + \omega(v)\}, \tag{1.3}$$

as the problem elements f and v are allowed to range over convergent

* Research supported in part by the National Science Foundation under Grant MCS 81-02684 at the University of Illinois at Urbana-Champaign.

0021-9045/82/080311-44$02.00/0

sequences and ε tends toward zero. We regard all problem constraints as being embodied in f, via the use of the possible function value $+\infty$.

Other work involving approximating sequences of optimization or variational problems, carried out in a spirit roughly similar to that followed here, can be found, for example, in Refs. [3–9, 13, 15–20, 24–27, 29–31, 34–37, 41, 44–46, 48–49]. In these sources one can find many further references to related work in a wide variety of areas of application, including approximation, nonlinear programming, stochastic optimization, control theory, free boundary problems, evolution equations, variational problems with obstacles, and others.

The various sequences involved will be indexed with the subscript α, ranging over the values 1, 2,... ∞. The convergence $v_\alpha \to v_\infty$ considered here with regard to problem $P(v)$ will usually be in the strong (i.e., norm) topology on V, although certain refinements will involve merely weak or weak* convergence.

A key issue is what notion to use for the convergence $f_\alpha \to f_\infty$. It turns out, perhaps surprisingly, that the notion most natural for the present work is not ordinary pointwise convergence of functions, but rather a distinctly different yet subtly related convergence notion, one arising from both geometric and technical considerations. It can be defined in terms of epigraphs, that is, the sets of the form

$$\operatorname{epi} f = \{(x, \mu) \in X \times R \mid f(x) \leqslant \mu\},$$

as follows: $f_\alpha \to f_\infty$ if and only if

$$w\text{-}\overline{\lim} \operatorname{epi} f_\alpha \subset \operatorname{epi} f_\infty \subset s\text{-}\underline{\lim} \operatorname{epi} f_\alpha. \tag{1.4}$$

Here, s-$\underline{\lim}$ and w-$\overline{\lim}$ denote the usual *limit inferior* and *limit superior* of a sequence of sets, except taken in the strong (i.e., norm) and weak topology, respectively. Thus,

$$s\text{-}\underline{\lim} \operatorname{epi} f_\alpha = \{s\text{-}\lim(x_\alpha, \mu_\alpha) \mid (x_\alpha, \mu_\alpha) \in \operatorname{epi} f_\alpha,\ \forall \alpha \in (\alpha)\}$$

and

$$w\text{-}\overline{\lim} \operatorname{epi} f_\alpha = \{w\text{-}\lim(x_\beta, \mu_\beta) \mid (x_\beta, \mu_\beta) \in \operatorname{epi} f_\beta,\ \forall \beta \in (\beta),\ \forall (\beta) \subset (\alpha)\},$$

where (α) denotes the sequence 1, 2,... (excluding ∞) and we write simply $(\beta) \subset (\alpha)$ to denote a subsequence (β) of (α). In words, (1.4) says that (a) each point of $\operatorname{epi} f_\infty$ is realizable as the strong limit of a sequence drawn from the $\operatorname{epi} f_\alpha$'s, and (b) $\operatorname{epi} f_\infty$ contains each weak subsequential limit of each sequence drawn from the $\operatorname{epi} f_\alpha$'s. Alternatively, (1.4) can be regarded as saying that the set-valued mapping $\alpha \to \operatorname{epi} f_\alpha$ is (a) strongly lower semicontinuous at $\alpha = \infty$, and (b) weakly upper semicontinuous at $\alpha = \infty$.

Mosco [35] was the first to consider this specific convergence notion (1.4), with its crucial distinction between weak and strong topologies, while the finite-dimensional case was considered a bit earlier by Wijsman [45, 46]. Salinetti and Wets [41] have given detailed study to comparing and contrasting the present notion $f_\alpha \to f_\infty$ with ordinary pointwise convergence (see also Marcellini [29]; Attouch [4, Prop. 1.19], [5, Prop. 1.7]; Denkowski [17]; and Dolecki *et al.* [20]).

Depending on the particular space X under consideration, variants of (1.4) may be appropriate, sometimes yielding refined results. When X is the dual of some other normed linear space V_0, for example, many of the results below admit variants/refinements in which the role of $w\text{-}\overline{\lim}\,\mathrm{epi}\, f_\alpha$ (cf. (1.4)) is played by $w^*\text{-}\overline{\lim}\,\mathrm{epi}\, f_\alpha$, involving the weak* topology induced on X by V_0. As another example, in a discussion involving spaces paired in duality, the strong topology in (1.4) would generally be replaced by the Mackey topology (cf. Joly [24, 25]).

The paper is organized as follows. Section 2 gives certain background dealing with $f_\alpha \to f_\infty$ and with convex analysis. In Sections 3–4 the general behavior of $\omega(v)$ and $\Omega(v, \varepsilon)$ is examined for convergent sequences of problems $P(v)$. For example, in Section 4, concerning the set $\Omega_\infty(v_\infty, 0)$ of exact solutions to the limit problem $P_\infty(v_\infty)$, Theorem 4 provides strong "necessary conditions" in terms of the sets $\Omega_\alpha(v_\alpha, 0)$ as $\alpha \to \infty$, while Theorem 5 provides quite weak "sufficient conditions" applying to nonconvex problems in terms of the sets $\Omega_\alpha(v_\alpha, \varepsilon_\alpha)$ as $\alpha \to \infty$. In Sections 5–6 the idea of sufficiency as embodied in Theorem 5 is explored in more detail, with several variants and refinements obtained. Among them, we point out in particular Theorem 10 of Section 6 and the remarks following it. In Section 7 we see that significant refinements of the earlier results are possible when X is finite dimensional. Throughout Sections 4–7, some of the results are also derived in dual form, yielding for nonconvex functions new technical properties of $f_\alpha \to f_\infty$ involving both function values and ε-subgradients.

2. Preliminaries

Throughout the paper, whenever a collection of functions $f_1, f_2, \ldots, f_\infty$ appears it is assumed that $f_\alpha : X \to [-\infty, +\infty]$ for each $\alpha = 1, 2, \ldots, \infty$, where X is some fixed real normed linear space having topological dual V. Additional hypotheses on X and/or the f_α's will be explicitly introduced as needed. The symbol B will always denote the unit ball centered at the origin.

Insight into the nature of the convergence defned by (1.4) is given by the following characterization in terms of function values.

LEMMA 1 (Mosco [35, Lemma 1.10]). (a) *One has* $\operatorname{epi} f_\infty \subset s\text{-}\underline{\lim}\, \operatorname{epi} f_\alpha$ *if and only if for every* x_∞ *there exists a sequence* (x_α) *such that*

$$x_\infty = s\text{-}\lim x_\alpha \qquad \textit{and} \qquad \overline{\lim}\, f_\alpha(x_\alpha) \leqslant f_\infty(x_\infty).$$

(b) *One has* $w\text{-}\overline{\lim}\, \operatorname{epi} f_\alpha \subset \operatorname{epi} f_\infty$ *if and only if*

$$f_\infty(x_\infty) \leqslant \underline{\lim}\, f_\beta(x_\beta)$$

holds whenever $(\beta) \subset (\alpha)$ *and* $x_\infty = w\text{-}\lim x_\beta$.

It is not hard to show that in Lemma 1(b) it is enough to take just the single, trivial subsequence $(\beta) = (\alpha)$.

COROLLARY. *If* $f_\alpha \to f_\infty$, *then for every* x_∞ *there exists a sequence* (x_α) *such that*

$$x_\infty = s\text{-}\lim x_\alpha \qquad \textit{and} \qquad f_\infty(x_\infty) = \lim f_\alpha(x_\alpha).$$

We recall next several notions from convex analysis. For further background on this one can consult, for example, Moreau [33], Brøndsted [11], Rockafellar [40], Laurent [28], or Ekeland and Temam [22].

For any function $f\colon X \to [-\infty, +\infty]$, the set

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$$

is the *effective domain* of f. We say f is *proper* provided f is never $-\infty$ and not identically $+\infty$, and that f is *convex* provided $\operatorname{epi} f$ is convex in $X \times R$.

Following Fenchel [23], we say that the *conjugate* of f (not assumed here to be convex or even proper) is the function $f^*\colon V \to [-\infty, +\infty]$ given by

$$f^*(v) = \sup_x \{\langle x, v\rangle - f(x)\}. \tag{2.1}$$

Immediately from (2.1), (1.2) one has

$$\omega(v) = -f^*(v). \tag{2.2}$$

One can consider also the *biconjugate* of f, which is the function f^{**} on X given by

$$f^{**}(x) = \sup_v \{\langle x, v\rangle - f^*(v)\}. \tag{2.3}$$

It is weakly lower semicontinuous and convex. One has f^* proper only if f is proper. On the other hand, f^* is never $-\infty$ when f is not identically $+\infty$, and f^* is not identically $+\infty$ provided f admits at least one weakly continuous affine minorant. In the latter case, f^{**} coincides with the largest weakly lower semicontinuous convex minorant of f. It follows that

$$f^{**} = f \tag{2.4}$$

when f is proper convex and weakly lower semicontinuous.

When X is a dual space, say the dual of V_0, then one can take the supremum in (2.3) just over V_0. Here, f^{**} is never $-\infty$ provided f admits at least one weak* continuous affine minorant, in which case f^{**} coincides with the largest weak* lower semicontinuous convex minorant of f. Therefore (2.4) holds when f is proper convex and weak* lower semicontinuous.

For any $f: X \to [-\infty, +\infty]$ and any $\varepsilon \in [0, +\infty)$, let us define the ε-*subdifferential* of f to be the multifunction $\partial_\varepsilon f: X \to V$ given by

$$v \in \partial_\varepsilon f(x) \quad \Leftrightarrow \quad f(x') \geqslant f(x) - \varepsilon + \langle x' - x, v \rangle, \forall x' \in X. \tag{2.5}$$

The image sets $\partial_\varepsilon f(x)$ are weak* closed and convex, although possibly empty. Using (1.2), (1.3), (2.1) and (2.2), one can check easily that, for any $x \in X$, $v \in V$ and $\varepsilon \in [0, +\infty)$, the following five conditions are pairwise equivalent:

$$x \in \Omega(v, \varepsilon), \tag{2.6}$$

$$\omega(v) \leqslant f(x) - \langle x, v \rangle \leqslant \varepsilon + \omega(v), \tag{2.7}$$

$$\inf_y \{f(y) - \langle y - x, v \rangle\} \leqslant f(x) \leqslant \varepsilon + \inf_y \{f(y) - \langle y - x, v \rangle\}, \tag{2.8}$$

$$f^*(v) - \varepsilon \leqslant \langle x, v \rangle - f(x) \leqslant f^*(v), \tag{2.9}$$

$$v \in \partial_\varepsilon f(x). \tag{2.10}$$

We shall use also the multifunction $\partial_\varepsilon^* f^*: V \to X$ defined by

$$x \in \partial_\varepsilon^* f^*(v) \quad \Leftrightarrow \quad f^*(v') \geqslant f^*(v) - \varepsilon + \langle x, v' - v \rangle, \forall v' \in V. \tag{2.11}$$

The image sets $\partial_\varepsilon^* f^*(v)$ are weakly closed and convex, and possibly empty. The inequality $f \geqslant f^{**}$ and characterization (2.9) yield that in general

$$v \in \partial_\varepsilon f(x) \Rightarrow x \in \partial_\varepsilon^* f^*(v), \tag{2.12}$$

while the converse implication holds provided $f(x) = f^{**}(x)$.

The biconjugate f^{**} on X defines a problem in the same way f defines $P(v)$. We say the *closed convex regularization* of $P(v)$ is the problem

$$\bar{P}(v): \text{minimize } f^{**}(x) - \langle x, v \rangle \text{ over } x \in X. \tag{2.13}$$

To this problem we associate an optimal value and approximate optimal solution sets just as in (1.2), (1.3):

$$\bar{\omega}(v) = \inf_x \{f^{**}(x) - \langle x, v\rangle\}, \tag{2.14}$$

$$\bar{\Omega}(v, \varepsilon) = \{x \in X \mid f^{**}(x) - \langle x, v\rangle \leqslant \varepsilon + \bar{\omega}(v)\}. \tag{2.15}$$

From $(f^{**})^* = f^*$ one obtains immediately that

$$\bar{\omega}(v) = \omega(v). \tag{2.16}$$

Since $\partial_\varepsilon f^{**}: X \to V$ is given by

$$v \in \partial_\varepsilon f^{**}(x) \quad \Leftrightarrow \quad f^{**}(x') \geqslant f^{**}(x) - \varepsilon + \langle x' - x, v\rangle, \forall x' \in X, \tag{2.17}$$

one can also check easily that the following five conditions are pairwise equivalent:

$$x \in \bar{\Omega}(v, \varepsilon), \tag{2.18}$$

$$\bar{\omega}(v) \leqslant f^{**}(x) - \langle x, v\rangle \leqslant \varepsilon + \bar{\omega}(v), \tag{2.19}$$

$$\inf_y \{f^{**}(y) - \langle y, v\rangle\} \leqslant f^{**}(x) \leqslant \varepsilon + \inf_y \{f^{**}(y) - \langle y - x, v\rangle\}, \tag{2.20}$$

$$f^*(v) - \varepsilon \leqslant \langle x, v\rangle - f^{**}(x) \leqslant f^*(v), \tag{2.21}$$

$$v \in \partial_\varepsilon f^{**}(x). \tag{2.22}$$

Finally, observe that when X is the dual of some other space, say $V_0 \subset V$, then the preceding three paragraphs still apply with

$$V,\ \partial_\varepsilon f,\ \text{weak}^*,\ \partial_\varepsilon^* f^*,\ \text{weakly},\ \partial_\varepsilon f^{**}$$

replaced everywhere, respectively, by

$$V_0,\ \partial_\varepsilon^* f,\ \text{weakly},\ \partial_\varepsilon f^*,\ \text{weak}^*,\ \partial_\varepsilon^* f^{**}.$$

We close this section by recalling two basic theorems concerning $f_\alpha \to f_\infty$. In view of (2.2) and the equivalences among (2.6)–(2.10), they go far toward establishing the appropriateness of definition (1.4) to our present study, at least in the convex case.

THEOREM 1. *Assume X is a reflexive Banach space and that each function $f_1, f_2, \ldots, f_\infty$ is proper convex and norm lower semicontinuous. Then one has $f_\alpha \to f_\infty$ if and only if $f_\alpha^* \to f_\infty^*$.*

This theorem was established by Wijsman [45, 46] in the finite-dimensional case. Independently, Walkup and Wets [43] established for reflexive Banach spaces a closely related result involving polar cones. Their result, incorporating a metric viewpoint, is in a sense stronger and, as was pointed out by R. T. Rockafellar, implies Theorem 1 (see [40, Theorem 14.4]). Theorem 1 per se was established for reflexive Banach spaces by Mosco [36] and Joly [24, 25]. Theorem 3 in Section 3 can be regarded as a nonconvex counterpart of Theorem 1.

For the second background theorem we adopt the notation

$$G(\partial_0 f) = \{(x, v) \in X \times V \mid x \in \partial_0 f(x)\}.$$

THEOREM 2. *Assume X is a Hilbert space and that each function $f_1, f_2, \ldots, f_\infty$ is proper convex and norm lower semicontinuous. Then one has $f_\alpha \to f_\infty$ if and only if*

(i) $w\text{-}\overline{\lim}\, G(\partial_0 f_\alpha) \subset G(\partial_0 f_\infty) \subset s\text{-}\underline{\lim}\, G(\partial_0 f_\alpha)$ *and*

(ii) *there exist pairs $(x_\alpha, v_\alpha) \in G(\partial_0 f_\alpha)$ for $\alpha = 1, 2, \ldots, \infty$ such that*

$$(x_\infty, v_\infty) = s\text{-}\lim (x_\alpha, v_\alpha) \qquad \textit{and} \qquad f_\infty(x_\infty) = \lim f_\alpha(x_\alpha).$$

This theorem is due to Attouch [3, 5], who also gives other characterizations in terms of resolvants and Yosida approximants (see also Brézis [10]). Matzeu [30] has proved a result for separable reflexive Banach spaces which is similar to Theorem 2 but with the roles of the weak and strong topologies intermixed; see also Boccardo and Marcellini [9]. The "only if" half of Theorem 2 is extended in a number of ways by Theorems 4 and 5 of Section 4.

3. BEHAVIOR OF $\omega(v)$ FOR CONVERGENT SEQUENCES OF PROBLEMS $P(v)$

To each f_α in a collection of functions $f_1, f_2, \ldots, f_\infty$ we associate an optimization problem of the type (1.1), denoted by $P_\alpha(\cdot)$, together with optimal value $\omega_\alpha(\cdot)$ an approximate solution sets $\Omega_\alpha(\cdot, \cdot)$ given by (1.2) and (1.3), respectively.

The following result has essentially been observed already by Salinetti and Wets [41, p. 223].

PROPOSITION 1. *Assume X is a reflexive Banach space and that $f_\alpha \to f_\infty$, where each function $f_1 f_2, \ldots, f_\infty$ is proper convex and norm lower semicontinuous. Then for every v_∞ there exists (v_α) such that*

$$v_\infty = s\text{-}\lim v_\alpha, \qquad \omega_\infty(v_\infty) = \lim \omega_\alpha(v_\alpha).$$

Proof. By Theorem 1, $f_\alpha^* \to f_\infty^*$. For any v we can thus apply the Corollary to Lemma 1 and appeal to (2.2).

We turn now to the nonconvex case. The following result will play a key role in several subsequent results.

LEMMA 2. *Assume that* epi $f_\infty \subset s\text{-}\underline{\lim}$ epi f_α. *Then one has*

$$\omega_\infty(v_\infty) \geqslant \overline{\lim}\, \omega_\beta(v_\beta) \tag{3.1}$$

whenever $(\beta) \subset (\alpha)$ *and* $v_\infty = w\text{-}\lim v_\beta$ *in* V. *If* X *is a Banach space, then* (3.1) *holds whenever* $v_\infty = w^*\text{-}\lim v_\beta$ *in* V. *If* X *is the dual of some other normed linear space* V_0, *then* (3.1) *holds whenever* $v_\infty = w\text{-}\lim v_\beta$ *in* V_0.

Proof. Let $(\beta) \subset (\alpha)$ and (v_β), v_∞ be given satisfying any of the alternate hypotheses. By (1.2), inequality (3.1) is equivalent to the condition

$$f_\infty(x) - \langle x, v_\infty \rangle \geqslant \overline{\lim}\, \inf_y \{ f_\beta(y) - \langle y, v_\beta \rangle \}, \qquad \forall x \in X.$$

Under any of the alternate hypotheses one has $\langle x, v_\infty \rangle = \lim \langle x, v_\beta \rangle$ for each $x \in X$ and also

$$\exists r < +\infty \text{ such that } \|v_\beta\| \leqslant r, \qquad \forall \beta \in (\beta) \tag{3.2}$$

(e.g., [47, pp. 120 and 125]). In particular, for (3.1) it suffices to prove that

$$f_\infty(x) \geqslant \overline{\lim}\, \inf_y \{ f_\beta(y) - \langle y - x, v_\beta \rangle \}, \qquad \forall x \in X.$$

Let $x \in X$. By Lemma 1(a), there exists (x_α) such that

$$x = s\text{-}\lim x_\alpha, \qquad \overline{\lim}\, f_\alpha(x_\alpha) \leqslant f_\infty(x).$$

From

$$f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle \geqslant \inf_y \{ f_\beta(y) - \langle y, v_\beta \rangle \}$$

and also (using (3.2))

$$\langle x - x_\beta, v_\beta \rangle \leqslant \|x - x_\beta\| \cdot \|v_\beta\| \leqslant r \|x_\beta - x\|,$$

we obtain

$$r \|x_\beta - x\| + f_\beta(x_\beta) \geqslant \inf_y \{ f_\beta(y) - \langle y - x, v_\beta \rangle \}$$

for any $\beta \in (\beta)$. Therefore

$$0 + f_\infty(x) \geqslant \lim r \|x_\beta - x\| + \overline{\lim}\, f_\beta(x_\beta) \geqslant \overline{\lim}\, \inf_y \{ f_\beta(y) - \langle y - x, v_\beta \rangle \},$$

as desired.

In the next result we employ a modification of the notation (1.2):

$$\forall r \in [0, \infty), \qquad \omega^r(v) = \inf_{\|x\| \leqslant r} \{f(x) - \langle x, v \rangle\}. \tag{3.3}$$

LEMMA 3. *Assume X is a reflexive Banach space and that $w\text{-}\overline{\lim}\ \operatorname{epi} f_\alpha \subset \operatorname{epi} f_\infty$, where f_∞ is not identically $+\infty$. Then*

$$\omega^r_\infty(v_\infty) \leqslant \underline{\lim}\ \omega^r_\alpha(v_\alpha) \tag{3.4}$$

holds whenever $r \in (\bar{r}, +\infty)$ and $v_\infty = s\text{-}\lim v_\alpha$, where

$$\bar{r} = \inf\{\rho \mid \exists x \in \operatorname{dom} f_\infty, \|x\| = \rho\}.$$

Proof. Note that $\operatorname{dom} f_\infty \neq \varnothing$ yields $\bar{r} < +\infty$. Now suppose (3.4) failed for some $r \in (\bar{r}, +\infty)$ and some $v_\infty = s\text{-}\lim v_\alpha$. Then $\omega^r_\infty(v_\infty) \in R$ and there exist $(\beta) \subset (\alpha)$ and $\varepsilon > 0$ satisfying

$$\omega^r_\beta(v_\beta) \leqslant \omega^r_\infty(v_\infty) - 3\varepsilon, \qquad \forall \beta \in (\beta).$$

Hence, there exists (x_β) such that

$$\|x_\beta\| \leqslant r \qquad \text{and} \qquad f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle \leqslant \omega^r_\infty(v_\infty) - 2\varepsilon, \qquad \forall \beta \in (\beta).$$

By reflexivity, there exist $(\gamma) \subset (\beta)$ and x_∞ such that $x_\infty = w\text{-}\lim x_\gamma$. Pick $\bar{\gamma}$ so that

$$|\langle x_\gamma, v_\gamma \rangle - \langle x_\infty, v_\infty \rangle| \leqslant \varepsilon, \qquad \forall \gamma \geqslant \bar{\gamma}.$$

Then for each $\gamma \geqslant \bar{\gamma}$ we have

$$\begin{aligned} f_\gamma(x_\gamma) &\leqslant \langle x_\gamma, v_\gamma \rangle + \omega^r_\infty(v_\infty) - 2\varepsilon \\ &\leqslant \langle x_\infty, v_\infty \rangle + \omega^r_\infty(v_\infty) - \varepsilon \\ &< \langle x_\infty, v_\infty \rangle + \omega^r_\infty(v_\infty) \\ &\leqslant f_\infty(x_\infty). \end{aligned}$$

Therefore $\underline{\lim}\ f_\gamma(x_\gamma) < f_\infty(x_\infty)$, where $x_\infty = w\text{-}\lim x_\gamma$. By Lemma 1(b), this contradicts our hypothesis. Thus, (3.4) holds whenever $r \in (\bar{r}, +\infty)$ and $v_\infty = s\text{-}\lim v_\alpha$.

THEOREM 3. *Assume that X is a reflexive Banach space and that $f_\alpha \to f_\infty$, where f_∞ is not identically $+\infty$. Suppose there exist $r < +\infty$ and $\bar{\alpha}$ such that $f_\alpha(x) = +\infty$ for all $\|x\| > r$ and $\alpha = \bar{\alpha}, \ldots, \infty$. Then $\omega_\alpha \to \omega_\infty$, and moreover,*

$$\omega_\infty(v_\infty) = \lim \omega_\alpha(v_\alpha) \tag{3.5}$$

holds whenever $v_\infty = s\text{-}\lim v_\alpha$. *More generally*, (3.5) *holds if* $v_\infty = s\text{-}\lim v_\alpha$ *and there exist* $r < +\infty$ *and* $\bar{\alpha}$ *such that* $\omega_\alpha(v_\alpha) = \omega_\alpha^r(v_\alpha)$ *for all* $\alpha = \bar{\alpha}, \ldots, \infty$.

Proof. Combine Lemmas 1, 2 and 3.

4. Behavior of $\Omega(v, \varepsilon)$ for Convergent Sequences of Problems $P(v)$

We begin by establishing for the convex case a strong "necessary condition" which must be satisfied by the exact solutions of the limit problem $P_\infty(v_\infty)$ in relation to the solutions of the approximating problems $P_\alpha(v_\alpha)$.

THEOREM 4. *Assume* X *is a reflexive Banach space and that* $f_\alpha \to f_\infty$, *where each function* $f_1, f_2, \ldots, f_\infty$ *is proper convex and norm lower semicontinuous. Then for every* v_∞ *and every* $x_\infty \in \Omega_\infty(v_\infty, 0)$, *there exist* (v_α) *and* (x_α) *satisfying*

$$v_\infty = s\text{-}\lim v_\alpha, \qquad \omega_\infty(v_\infty) = \lim \omega_\alpha(v_\alpha),$$
$$x_\infty = s\text{-}\lim x_\alpha, \qquad x_\alpha \in \Omega_\alpha(v_\alpha, 0).$$

Proof. By (2.2) and the equivalence (2.6) $\Leftrightarrow$ (2.10) for $\varepsilon = 0$, it suffices to establish the following: If $(x_\infty, v_\infty) \in G(\partial_0 f_\infty)$, then there exists a sequence of pairs $(x_\alpha, v_\alpha) \in G(\partial_0 f_\alpha)$ such that

$$(x_\infty, v_\infty) = s\text{-}\lim(x_\alpha, v_\alpha), \tag{4.1}$$
$$f_\infty^*(v_\infty) = \lim f_\alpha^*(v_\alpha). \tag{4.2}$$

We argue by contradiction. Suppose (4.1) fails, i.e., that $(x_\infty, v_\infty) \in G(\partial_0 f_\infty)$ and there exists an $\varepsilon > 0$ for which

$$G(\partial_0 f_\alpha) \cap [(x_\infty, v_\infty) + 2\varepsilon B] = \varnothing \tag{4.3}$$

occurs for infinitely many α's, say on a subsequence $(\beta) \subset (\alpha)$. Here, $B = \{(x, v) \in X \times V \mid \|x\|^2 + \|v\|^2 \leqslant 1\}$. By Theorem 1 we can apply Lemma 1(a) in V as well as in X to obtain sequences (x_α) and (v_α) satisfying

$$x_\infty = s\text{-}\lim x_\alpha, \qquad \overline{\lim}\, f_\alpha(x_\alpha) \leqslant f_\infty(x_\infty), \tag{4.4}$$
$$v_\infty = s\text{-}\lim v_\alpha, \qquad \overline{\lim}\, f_\alpha^*(v_\alpha) \leqslant f_\infty^*(v_\infty). \tag{4.5}$$

We now claim there exists $\bar{\beta}$ such that

$$f_\beta(x_\beta) + f_\beta^*(v_\beta) - \langle x_\beta, v_\beta \rangle > \lambda, \qquad \forall \beta \geqslant \bar{\beta}, \tag{4.6}$$

where $\lambda = (\varepsilon^2)/2$. Using (4.4) and (4.5), pick $\bar{\beta}$ so that

$$(x_\beta, v_\beta) \in (x_\infty, v_\infty) + \varepsilon B, \qquad \forall \beta \geqslant \bar{\beta}. \tag{4.7}$$

Now let $\beta \geqslant \bar{\beta}$ be given, and suppose that

$$f_\beta(x_\beta) + f_\beta^*(v_\beta) - \langle x_\beta, v_\beta \rangle \leqslant \lambda. \tag{4.8}$$

This means that $v_\beta \in \partial_\lambda f_\beta(x_\beta)$. Hence, the lemma of Brøndsted and Rockafellar [12] implies that there exists a pair $(x, v) \in G(\partial_0 f_\beta)$ satisfying

$$\|x - x_\beta\| \leqslant \sqrt{\lambda}, \qquad \|v - v_\beta\| \leqslant \sqrt{\lambda}.$$

These imply $(x, v) \in (x_\beta, v_\beta) + \varepsilon B$, which combines with (4.7) to yield $(x, v) \in (x_\infty, v_\infty) + 2\varepsilon B$. Since also $(x, v) \in G(\partial_0 f_\beta)$, this contradicts (4.3). We conclude that (4.8) fails for each $\beta \geqslant \bar{\beta}$, i.e., that (4.6) holds. Now combining (4.6) with (4.4) and (4.5), we obtain

$$\begin{aligned} f_\infty(x_\infty) + f_\infty^*(v_\infty) &\geqslant \overline{\lim}\, f_\alpha(x_\alpha) + \overline{\lim}\, f_\alpha^*(v_\alpha) \\ &\geqslant \overline{\lim}(f_\alpha(x_\alpha) + f_\alpha^*(v_\alpha)) \\ &\geqslant \overline{\lim}(f_\beta(x_\beta) + f_\beta^*(v_\beta)) \\ &\geqslant \lambda + \overline{\lim}\langle x_\beta, v_\beta \rangle \\ &= \lambda + \langle x_\infty, v_\infty \rangle. \end{aligned}$$

But this conflicts with $f_\infty(x_\infty) + f_\infty^*(v_\infty) \leqslant \langle x_\infty, v_\infty \rangle$, which follows from the assumption $(x_\infty, v_\infty) \in G(\partial_0 f_\infty)$. This contradiction proves there exists a sequence of pairs $(x_\alpha, v_\alpha) \in G(\partial_0 f_\alpha)$ such that (4.1) holds. We now show that (4.2) also holds for any such sequence of pairs. By Theorem 1, we can apply Lemma 1(b) in V to obtain

$$f_\infty^*(v_\infty) \leqslant \underline{\lim}\, f_\alpha^*(v_\alpha) \tag{4.9}$$

and Lemma 1(a) in V to obtain a sequence (v'_α) satisfying

$$v_\infty = s\text{-}\lim v'_\alpha, \qquad \overline{\lim}\, f_\alpha^*(v'_\alpha) \leqslant f_\infty^*(v_\infty). \tag{4.10}$$

For each α, $(x_\alpha, v_\alpha) \in G(\partial_0 f_\alpha)$ yields

$$f_\alpha^*(v'_\alpha) \geqslant f_\alpha^*(v_\alpha) + \langle x_\alpha, v'_\alpha - v_\alpha \rangle.$$

Taking the limit superior throughout this inequality yields, in view of $\lim\langle x_\alpha, v'_\alpha - v_\alpha \rangle = \langle x_\infty, v_\infty - v_\infty \rangle = 0$, the inequality

$$\overline{\lim}\, f_\alpha^*(v'_\alpha) \geqslant \overline{\lim}(\langle x_\alpha, v'_\alpha - v_\alpha \rangle + f_\alpha^*(v_\alpha)) = \overline{\lim}\, f_\alpha^*(v_\alpha). \tag{4.11}$$

Combining (4.11), (4.10), (4.9) yields (4.2), completing the proof.

Happily, it turns out that an extremely weak "sufficient condition" applies quite broadly to the exact solutions of $P_\infty(v_\infty)$ in relation to the approximate solutions of the approximating problems $P_\alpha(v_\alpha)$. Any real normed linear space will do, and the f_α's need not be convex or even lower semicontinuous.

THEOREM 5. *Assume $f_\alpha \to f_\infty$. Let $(\beta) \subset (\alpha)$ and corresponding sequences (x_β), (v_β), (ε_β) satisfy the conditions*

$$x_\infty = w\text{-}\lim x_\beta, \qquad v_\infty = w\text{-}\lim v_\beta, \tag{4.12}$$

$$\langle x_\infty, v_\infty \rangle \geqslant \overline{\lim} \langle x_\beta, v_\beta \rangle, \tag{4.13}$$

$$x_\beta \in \Omega_\beta(v_\beta, \varepsilon_\beta), \qquad 0 \leqslant \varepsilon_\beta \to 0.$$

Then $x_\infty \in \Omega_\infty(v_\infty, 0)$, and furthermore

$$\lim(f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle) = \lim \omega_\beta(v_\beta) = \omega_\infty(v_\infty), \tag{4.14}$$

$$\lim \inf_x \{f_\beta(x) - \langle x - x_\beta, v_\beta \rangle\} = \lim f_\beta(x_\beta) = f_\infty(x_\infty), \tag{4.15}$$

where the limits are finite if f_∞ is proper. If X is a Banach space, then in (4.12) *the weak topology on V can be replaced by the weak* topology. If X is the dual of some other normed linear space V_0, then in* (4.12) *and the hypothesis $f_\alpha \to f_\infty$ (see* (1.4)) *the weak topology on X can be replaced by the weak* topology induced by V_0, provided $\{v_\beta \mid \beta = 1,..., \infty\} \subset V_0$. In any case, the technical condition* (4.13) *is fulfilled automatically whenever either of the limits in* (4.12) *actually occurs in the norm topology.*

Proof. We have that

$$\begin{aligned}
\overline{\lim}\, \omega_\beta(v_\beta) &\leqslant \overline{\lim}(f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle) \\
&\leqslant \overline{\lim}(\varepsilon_\beta + \omega_\beta(v_\beta)) \\
&= \overline{\lim}\, \omega_\beta(v_\beta) \\
&\leqslant \omega_\infty(v_\infty) && (4.16) \\
&\leqslant f_\infty(x_\infty) - \langle x_\infty, v_\infty \rangle \\
&\leqslant \underline{\lim}\, f_\beta(x_\beta) + \underline{\lim}(-\langle x_\beta, v_\beta \rangle) && (4.17) \\
&\leqslant \underline{\lim}(f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle) \\
&\leqslant \underline{\lim}(\varepsilon_\beta + \omega_\beta(v_\beta)) \\
&= \underline{\lim}\, \omega_\beta(v_\beta),
\end{aligned}$$

so equality holds throughout and the common value is the quantity

$$\omega_\infty(v_\infty) = f_\infty(x_\infty) - \langle x_\infty, v_\infty \rangle. \tag{4.18}$$

Here, (4.16) follows from Lemma 2, and (4.17) follows from Lemma 1(b) and (4.13). This establishes (4.14). We also have that

$$\begin{aligned}
\overline{\lim}(\omega_\beta(v_\beta) + \langle x_\beta, v_\beta \rangle) &\leqslant \overline{\lim}\, f_\beta(x_\beta) \\
&\leqslant \overline{\lim}\langle x_\beta, v_\beta \rangle + \overline{\lim}(f_\beta(x_\beta) - \langle x_\beta, v_\beta \rangle) \\
&\leqslant \langle x_\infty, v_\infty \rangle + (f_\infty(x_\infty) - \langle x_\infty, v_\infty \rangle) \qquad (4.19) \\
&= f_\infty(x_\infty) \\
&\leqslant \underline{\lim}\, f_\beta(x_\beta) \qquad (4.20) \\
&\leqslant \underline{\lim}(\varepsilon_\beta + \omega_\beta(v_\beta) + \langle x_\beta, v_\beta \rangle) \\
&= \underline{\lim}(\omega_\beta(v_\beta) + \langle x_\beta, v_\beta \rangle),
\end{aligned}$$

so equality holds throughout. Here, (4.19) follows from (4.13) and (4.14) (actually, just the first part of the string of inequalities leading to (4.14)), while (4.20) follows from Lemma 1(b). This establishes (4.15). Finally, (4.18) shows $x_\infty \in \Omega_\infty(v_\infty, 0)$ and that the common values in (4.14), (4.15) are finite if f_∞ is proper. The refinements involving alternate topologies follow from the preceding proof, by appealing to the refinements in Lemma 2 and to the obvious weak* variant of Lemma 1(b). The remark concerning (4.13) is elementary (e.g., [47, pp. 120 and 125]). This concludes the proof.

For the convex case in Hilbert spaces, the technical condition (4.13) can be avoided en route to obtaining $x_\infty \in \Omega_\infty(v_\infty, 0)$. The special case of this result in which the tolerances ε_β are identically zero (and $v_\infty = 0$) has been observed already by Wets [44, p. 400].

PROPOSITION 2. *Assume X is a Hilbert space and that $f_\alpha \to f_\infty$, where each function $f_1, f_2, \ldots, f_\infty$ is proper convex and norm lower semicontinuous. Let $(\beta) \subset (\alpha)$ and corresponding sequences (x_β), (v_β), (ε_β) satisfy the conditions*

$$x_\infty = w\text{-}\lim x_\beta, \qquad v_\infty = w\text{-}\lim v_\beta,$$
$$x_\beta \in \Omega_\beta(v_\beta, \varepsilon_\beta), \qquad 0 \leqslant \varepsilon_\beta \to 0.$$

Then $x_\infty \in \Omega_\infty(v_\infty, 0)$.

Proof. First, suppose each $\varepsilon_\beta = 0$. Then $(x_\beta, v_\beta) \in G(\partial_0 f_\beta)$ for each $\beta \in (\beta)$. Since $(x_\infty, v_\infty) = w\text{-}\lim(x_\beta, v_\beta)$, the left-hand inclusion in the "only if" half of Theorem 2 yields $(x_\infty, v_\infty) \in G(\partial_0 f_\infty)$, i.e., $x_\infty \in \Omega_\infty(v_\infty, 0)$. For the general case $0 \leqslant \varepsilon_\beta \to 0$, we appeal to the Brønsted–Rockafellar lemma

[12] for each $\beta \in (\beta)$. From $v_\beta \in \partial_{\varepsilon_\beta} f_\beta(x_\beta)$ this yields pairs $(\bar{x}_\beta, \bar{v}_\beta) \in G(\partial_0 f_\beta)$ satisfying

$$\|\bar{x}_\beta - x_\beta\| \leqslant \sqrt{\varepsilon_\beta}, \qquad \|\bar{v}_\beta - v_\beta\| \leqslant \sqrt{\varepsilon_\beta}.$$

Since $(x_\infty, v_\infty) = w\text{-lim}(x_\beta, v_\beta)$ and $0 = \lim \varepsilon_\beta$, it is routine to deduce $(x_\infty, v_\infty) = w\text{-lim}(\bar{x}_\beta, \bar{v}_\beta)$. The "only if" half of Theorem 2 can now be applied just as before, but to the pairs $(\bar{x}_\beta, \bar{v}_\beta)$ in place of (x_β, v_β).

Observe that Theorem 4 can be paraphrased as asserting (for the convex case in reflexive Banach spaces) that

$$\Omega_\infty(v_\infty, 0) \subset \bigcup \{s\text{-}\underline{\lim}\, \Omega_\alpha(v_\alpha, 0) \mid v_\infty = s\text{-lim}\, v_\alpha\}, \tag{4.21}$$

plus associated limit information about optimal values. Correspondingly, Theorem 5 can be paraphrased as asserting (for the nonconvex case in general normed linear spaces), in particular, that

$$\bigcup \{w\text{-}\overline{\lim}\, \Omega_\alpha(v_\alpha, \varepsilon_\alpha) \mid v_\infty = s\text{-lim}\, v_\alpha, 0 \leqslant \varepsilon_\alpha \to 0\} \subset \Omega_\infty(v_\infty, 0), \tag{4.22}$$

plus associated limit information about optimal values.

The next two sections explore further the type of sufficiency criterion for optimality exemplified by (4.22) and Theorem 5, including the issue of guaranteeing

$$\varnothing \neq w\text{-}\overline{\lim}\, \Omega_\alpha(v_\alpha, \varepsilon_\alpha). \tag{4.23}$$

5. Further Sufficiency Criteria

We begin with a sharpened form of Theorem 5 ensuring the existence of appropriate cluster points (cf. (4.23)).

Theorem 6. *Assume that X is a reflexive Banach space and that $f_\alpha \to f_\infty$. Let (x_α), (v_α), v_∞, (ε_α) satisfy*

$$x_\alpha \in \Omega_\alpha(v_\alpha, \varepsilon_\alpha), \qquad v_\infty = s\text{-lim}\, v_\alpha, \qquad 0 \leqslant \varepsilon_\alpha \to 0,$$

and assume there exist $r < +\infty$ and $\bar{a}$ such that $\|x_\alpha\| \leqslant r$ for all $a \geqslant \bar{a}$. Then

$$\lim(f_\alpha(x_\alpha) - \langle x_\alpha, v_\alpha\rangle) = \lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty), \tag{5.1}$$

and there exist $(\beta) \subset (\alpha)$ and x_∞ such that

$$w\text{-lim}\, x_\beta = x_\infty \in \Omega_\infty(v_\infty, 0), \tag{5.2}$$

$$\lim \inf_x \{f_\beta(x) - \langle x - x_\beta, v_\beta\rangle\} = \lim f_\beta(x_\beta) = f_\infty(x_\infty). \tag{5.3}$$

If $\langle x_\infty, v_\infty\rangle = \lim\langle x_\alpha, v_\alpha\rangle$, *then in* (5.3) *the* β*'s can be replaced by* α*'s. If* f_∞ *is proper, the limits in* (5.1), (5.3) *are finite.*

Proof. By reflexivity, the assumed bound on $\|x_\alpha\|$ implies the existence of $(\beta)\subset(\alpha)$ and x_∞ such that $x_\infty = w\text{-}\lim x_\beta$. Since condition (4.13) is fulfilled for (β), Theorem 5 immediately yields (5.2), (5.3) and (4.14), as well as the finiteness assertion once we have strengthened (4.14) to (5.1). Toward this end, suppose that $\underline{\lim}\,\omega_\alpha(v_\alpha) < \omega_\infty(v_\infty)$. Then there exist $\xi < \omega_\infty(v_\infty)$ and $(\gamma)\subset(\alpha)$ such that $\omega_\gamma(v_\gamma)\leqslant\xi$ for all $\gamma\in(\gamma)$. Since clearly $f_\gamma\to f_\infty$ (e.g., [35, p. 521]) and the x_γ's are norm bounded, part of the present theorem already established (specifically, the second equation of (4.14)) applies to yield a further subsequence $(\delta)\subset(\gamma)$ such that $\omega_\infty(v_\infty)\leqslant\overline{\lim}\,\omega_\delta(v_\delta)$. This results in the absurdity $\omega_\infty(v_\infty)\leqslant\xi<\omega_\infty(v_\infty)$, and thus shows that $\omega_\infty(v_\infty)\leqslant\underline{\lim}\,\omega_\alpha(v_\alpha)$. The same argument can be made, with all inequalities reversed and limits superior and inferior interchanged, to yield that $\omega_\infty(v_\infty)\geqslant\overline{\lim}\,\omega_\alpha(v_\alpha)$. It follows that

$$\begin{aligned}\overline{\lim}(f_\alpha(x_\alpha)-\langle x_\alpha, v_\alpha\rangle) &\leqslant \overline{\lim}\,\varepsilon_\alpha + \overline{\lim}\,\omega_\alpha(v_\alpha)\\ &\leqslant \omega_\infty(v_\infty)\\ &\leqslant \underline{\lim}\,\omega_\alpha(v_\alpha)\\ &\leqslant \underline{\lim}(f_\alpha(x_\alpha)-\langle x_\alpha, v_\alpha\rangle),\end{aligned}$$

which establishes (5.1). Finally, suppose $\langle x_\infty, v_\infty\rangle = \lim\langle x_\alpha, v_\alpha\rangle$. Using this, together with (5.1) and (5.2), we obtain

$$\begin{aligned}\underline{\lim}(\omega_\alpha(v_\alpha)+\langle x_\alpha, v_\alpha\rangle) &\geqslant \underline{\lim}(f_\alpha(x_\alpha)-\varepsilon_\alpha)\\ &= \underline{\lim}\, f_\alpha(x_\alpha)\\ &\geqslant \underline{\lim}(f_\alpha(x_\alpha)-\langle x_\alpha, v_\alpha\rangle)+\underline{\lim}\langle x_\alpha, v_\alpha\rangle\\ &\geqslant \omega_\infty(v_\infty)+\langle x_\infty, v_\infty\rangle\\ &= f_\infty(x_\infty),\end{aligned}$$

as well as the analogous estimates with inequalities reversed and limits superior (and the ε_α's suppressed). This establishes the refinement of (5.3) and completes the proof.

The next result is dual to Theorem 6. (Recall the equivalences among (2.6)–(2.10).)

THEOREM 7. *Assume that* X *is a reflexive Banach space and that* $f_\alpha\to f_\infty$. *Let* (x_α), x_∞, (v_α), (ε_α) *satisfy*

$$v_\alpha\in\partial_{\varepsilon_\alpha} f_\alpha(x_\alpha), \qquad x_\infty = s\text{-}\lim x_\alpha, \qquad 0\leqslant\varepsilon_\alpha\to 0,$$

and assume there exist $r < +\infty$ *and* $\bar{\alpha}$ *such that* $\|v_\alpha\| \leqslant r$ *for all* $\alpha \geqslant \bar{\alpha}$. *Then*

$$f_\infty(x_\infty) = \lim f_\alpha(x_\alpha) = \lim \inf_x \{f_\alpha(x) - \langle x - x_\alpha, v_\alpha\rangle\}, \tag{5.4}$$

and there exist $(\beta) \subset (\alpha)$ *and* v_∞ *such that*

$$w\text{-}\lim v_\beta = v_\infty \in \partial_0 f_\infty(x_\infty), \tag{5.5}$$

$$f_\infty^*(v_\infty) = \lim f_\beta^*(v_\beta) = \lim(\langle x_\beta, v_\beta\rangle - f_\beta(x)). \tag{5.6}$$

If $\langle x_\infty, v_\infty\rangle = \lim\langle x_\alpha, v_\alpha\rangle$, *then in* (5.6) *the* β*'s can be replaced by* α*'s. If* f_∞ *is proper, the limits in* (5.4), (5.6) *are finite.*

Proof. The first part of the argument is a straightforward reworking of the proof of Theorem 6, up to the point at which Theorem 5 yields (5.4), (5.6) and (4.15). From there, the details concerning replacing β's by α's can be a little tricky. One first obtains $f_\infty(x_\infty) \leqslant \underline{\lim}\, f_\alpha(x_\alpha)$ and $f_\infty(x_\infty) \geqslant \overline{\lim}\, f_\alpha(x_\alpha)$, in each case using a *reductio ad absurdum* argument based on applying the second equality of (4.15) to a subsequence $f_\gamma \to f_\infty$. Using these inequalities, one obtains

$$\begin{aligned}
f_\infty(x_\infty) &\leqslant \underline{\lim}\, f_\alpha(x_\alpha) \\
&\leqslant \underline{\lim}(\varepsilon_\alpha + \inf_x \{f_\alpha(x) - \langle x - x_\alpha, v_\alpha\rangle\}) \\
&= \underline{\lim} \inf_x \{f_\alpha(x) - \langle x - x_\alpha, v_\alpha\rangle\} \\
&\leqslant \overline{\lim}\, f_\alpha(x_\alpha) \\
&\leqslant f_\infty(x_\infty),
\end{aligned}$$

establishing (5.4). Finally, from $\langle x_\infty, v_\infty\rangle = \lim\langle x_\alpha, v_\alpha\rangle$, together with (5.4) and (5.5), one obtains

$$\begin{aligned}
\underline{\lim}(\langle x_\alpha, v_\alpha\rangle - f_\alpha(x_\alpha)) &\geqslant \underline{\lim}(f_\alpha^*(v_\alpha) - \varepsilon_\alpha) \\
&= \underline{\lim}\, f_\alpha^*(v_\alpha) \\
&\geqslant \underline{\lim}\langle x_\alpha, v_\alpha\rangle + \underline{\lim}(f_\alpha^*(v_\alpha) - \langle x_\alpha, v_\alpha\rangle) \\
&\geqslant \langle x_\infty, v_\infty\rangle - f_\infty(x_\infty) \\
&= f_\infty^*(v_\infty),
\end{aligned}$$

as well as the analogous estimates with inequalities reversed and limits superior (and the ε_α's suppressed). This establishes the refinement of (5.6).

The next result provides a convenient criterion for exploiting Theorem 6.

THEOREM 8. *Assume that X is a reflexive Banach space and that $f_\alpha \to f_\infty$, where f_∞ is not identically $+\infty$. Let $\bar{v}_\infty$ be such that there exists a proper convex norm lower semicontinuous function k on X having the following two properties*: (i) *$k \leqslant f_\alpha$ for all α sufficiently large (excluding $\alpha = \infty$)*; (ii) *there exists $\gamma > \inf\{k - \langle \cdot, \bar{v}_\infty \rangle\}$ such that $\{x \in X \mid k(x) - \langle x, \bar{v}_\infty \rangle \leqslant \gamma\}$ is weakly compact. Then there exists $\mu > 0$ such that for any v_∞, (v_α), (x_α), (ε_α) satisfying*

$$\|v_\infty - \bar{v}_\infty\| < \mu, \qquad v_\infty = s\text{-}\lim v_\alpha,$$
$$x_\alpha \in \Omega_\alpha(v_\alpha, \varepsilon_\alpha), \qquad 0 \leqslant \varepsilon_\alpha \to 0, \tag{5.7}$$

one has

$$\lim(f_\alpha(x_\alpha) - \langle x_\alpha, v_\alpha \rangle) = \lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R \tag{5.8}$$

and also the existence of $(\beta) \subset (\alpha)$ and x_∞ such that

$$w\text{-}\lim x_\beta = x_\infty \in \Omega_\infty(v_\infty, 0), \tag{5.9}$$
$$\lim \inf_x \{f_\beta(x) - \langle x - x_\beta, v_\beta \rangle\} = \lim f_\beta(x_\beta) = f_\infty(x_\infty) \in R. \tag{5.10}$$

If $\langle x_\infty, v_\infty \rangle = \lim \langle x_\alpha, v_\alpha \rangle$, then in (5.10) *the β's can be replaced by α's. In particular, one has*

$$\varnothing \neq w\text{-}\overline{\lim}\, \Omega_\alpha(v_\alpha, \varepsilon_\alpha) \subset \Omega_\infty(v_\infty, 0), \tag{5.11}$$
$$\lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R \tag{5.12}$$

whenever $v_\infty = s\text{-}\lim v_\alpha$, $\|v_\infty - \bar{v}_\infty\| < \mu$, $0 < \varepsilon_\alpha \to 0$.

Proof. By the Moreau–Rockafellar theorem [32, 33, 38], property (ii) implies k^* is bounded above on $\bar{v}_\infty + \mu B$ for some $\mu > 0$. Now consider any $\varepsilon > 0$ and $v_\infty = s\text{-}\lim v_\alpha$, where $\|v_\infty - \bar{v}_\infty\| < \mu$. We have k^* bounded above on a norm neighborhood of v_∞, so k^* is norm upper semicontinuous at v_∞. Moreover, results of Moreau [32, 33] (see also Asplund and Rockafellar [2, Theorem 2]) imply that for any $\lambda \in [0, +\infty)$ there exists $\tilde{\mu} > 0$ such that the set

$$T = \bigcup \{\partial_\lambda k^*(v) \mid v \in v_\infty + \tilde{\mu} B\}$$

is Mackey equicontinuous, hence norm bounded by some $r < +\infty$. Now take λ to be

$$\lambda = 3\varepsilon + k^*(v_\infty) + \omega_\infty(v_\infty).$$

Notice that $\lambda \in (0, +\infty)$. (Indeed,

$$-\infty < -k^*(v_\infty) \leqslant -f^*_\infty(v_\infty) = \omega_\infty(v_\infty) < +\infty$$

follows from

$$k(y) \leqslant \underline{\lim}\, k(y_\alpha) \leqslant \overline{\lim}\, f_\alpha(y_\alpha) \leqslant f(y), \qquad \forall y \in X,$$

which itself follows by property (i) and Lemma 1(a).) Using upper semicontinuity, choose $\bar{\mu} \in (0, \tilde{\mu}]$ so that $k^*(v) \leqslant \varepsilon + k^*(v_\infty)$ for all $v \in v_\infty + \bar{\mu}B$. Then pick $\bar{\alpha}$ so that $v_\alpha \in v_\infty + \bar{\mu}B$ and $k \leqslant f_\alpha$ for all $\alpha \geqslant \bar{\alpha}$, and finally, using Lemma 2, pick $\tilde{\alpha} \geqslant \bar{\alpha}$ so that $\omega_\alpha(v_\alpha) \leqslant \varepsilon + \omega_\infty(v_\infty)$ for all $\alpha \geqslant \alpha$. Then for any $\alpha \geqslant \tilde{\alpha}$ and any $x \in \partial^*_\varepsilon f^*_\alpha(v_\alpha) \supset \Omega_\alpha(v_\alpha, \varepsilon)$, the estimates

$$\begin{aligned}
k(x) - \langle x, v_\alpha \rangle &\leqslant f^{**}_\alpha(x) - \langle x, v_\alpha \rangle && \text{(by (i))}\\
&\leqslant \varepsilon + \omega_\alpha(v_\alpha) && (\text{by } x \in \partial^*_\varepsilon f^*_\alpha(v_\alpha))\\
&\leqslant \varepsilon + \varepsilon + \omega_\infty(v_\infty) && (\text{by } \tilde{\alpha})\\
&= \lambda - \varepsilon - k^*(v_\infty) && (\text{by } \lambda)\\
&\leqslant \lambda - \varepsilon + \varepsilon - k^*(v_\alpha) && (\text{by } \bar{\mu}, \bar{\alpha})
\end{aligned}$$

imply that $x \in \partial_\lambda k^*(v_\alpha) \subset T$ (by $\tilde{\mu}, \bar{\alpha}$). Thus, whenever $v_\infty = s\text{-lim}\, v_\alpha$, $\|v_\infty - \bar{v}_\infty\| < \mu$, $\varepsilon > 0$ there exist $\lambda > 0$, $r < +\infty$, $\tilde{\alpha}$ satisfying

$$\partial^*_\varepsilon f^*_\alpha(v_\alpha) \subset \partial_\lambda k^*(v_\alpha) \subset \{x \in X \mid \|x\| \leqslant r\}, \qquad \forall \alpha \geqslant \tilde{\alpha}. \tag{5.13}$$

Suppose now that (x_α), (ε_α) satisfy (5.7). By (5.13), there exists $r < +\infty$ such that $\|x_\alpha\| \leqslant r$ for all α sufficiently large, so that Theorem 6 applies. From (5.1), (5.2) and (5.3) follow (5.8), (5.9) and (5.10), respectively, as well as the refinement of (5.10). The second part of (5.11) follows from Theorem 5. The existence part of (5.11), as well as (5.12), will follow from (5.9) and (5.8), respectively, provided a sequence (x_α) satisfying (5.7) can be chosen. This is indeed possible when $0 < \varepsilon_\alpha \to 0$, due to the fact that, for all α sufficiently large, $-\infty < -k^*(v_\alpha) \leqslant \omega_\alpha(v_\alpha)$ (from property (i) and properties of k) and $\omega_\alpha(v_\alpha) < +\infty$ (by Lemma 2 and $\omega_\infty(v_\infty) < +\infty$). This concludes the proof.

From Theorem 8 we can deduce information concerning the behavior of the directional derivatives of $\omega_\alpha(\cdot)$ as $\alpha \to \infty$. The result actually handles approximate directional derivatives as well. Since $\omega(\cdot)$ is concave (cf. (2.2)), we define these whenever $\omega(v)$ is finite by means of

$$\omega'_\varepsilon(v; z) = \sup_{\tau > 0} \left\{ \frac{\omega(v + \tau z) - \omega(v) - \varepsilon}{\tau} \right\}, \qquad \forall v, z \in V, \tag{5.14}$$

for any $\varepsilon \in [0, +\infty)$. When $\varepsilon = 0$, this is just

$$\omega_0'(v; z) = \lim_{\tau \downarrow 0} \frac{\omega(v + \tau z) - \omega(v)}{\tau}, \tag{5.15}$$

which is of traditional interest due to the "marginal rate of change" data it conveys. We emphasize the natural importance of the quantities (5.14) for strictly positive values of ε also. This is due to the fact (cf. Rockafellar [39, p. 504], [40, p. 220]; and Moreau [33, p. 67]) that

$$\sup_{\tau > 0} \left\{ \frac{\omega(v + \tau z) - \omega(v) - \varepsilon}{\tau} \right\} = \inf\{\langle x, z \rangle \mid x \in \bar{\Omega}(v, \varepsilon)\} \tag{5.16}$$

whenever $\varepsilon \in [0, +\infty)$ and $z \in V$ (recall (2.13)–(2.15)).

COROLLARY. *Under the hypotheses of Theorem* 8, *the approximate directional derivative functions* $(\omega_\alpha)_\varepsilon'(v; z)$ *defined as in* (5.14) *satisfy*

$$(\omega_\infty)_0'(v_\infty; z_\infty) \leqslant \underline{\lim}(\omega_\alpha)_{\varepsilon_\alpha}'(v_\alpha; z_\alpha) \tag{5.17}$$

whenever $v_\infty = s\text{-}\lim v_\alpha$, $\|v_\infty - \bar{v}_\infty\| < \mu$, $0 \leqslant \varepsilon_\alpha \to 0$, $z_\infty = s\text{-}\lim z_\alpha$.

Proof. Let v_∞, (v_α), z_∞, (z_α), (ε_α) be given as described. Since ω_∞ is concave and finite at v_∞, the difference quotient $\tau^{-1}[\omega_\infty(v_\infty + \tau z_\infty) - \omega_\infty(v_\infty)]$ is nondecreasing as $\tau \downarrow 0$ (e.g., [33, p. 64] or [40, p. 214]). Hence, given any $\sigma < (\omega_\infty)_0'(v_\infty; z_\infty)$, there exists a $\tau > 0$ such that

$$\|(v_\infty + \tau z_\infty) - \bar{v}_\infty\| < \mu, \qquad \sigma < \tau^{-1}[\omega_\infty(v_\infty + \tau z_\infty) - \omega_\infty(v_\infty)].$$

Since $v_\infty = s\text{-}\lim v_\alpha$ and $z_\infty = s\text{-}\lim z_\alpha$, assertion (5.12) of Theorem 8 implies

$$\omega_\infty(v_\infty + \tau z_\infty) = \lim \omega_\alpha(v_\alpha + \tau z_\alpha), \qquad \omega_\infty(v_\infty) = \lim \omega_\alpha(v_\alpha).$$

Since $0 = \lim \varepsilon_\alpha$, it follows that for all sufficiently large α one has

$$\sigma < \tau^{-1}[\omega_\alpha(v_\alpha + \tau z_\alpha) - \omega_\alpha(v_\alpha) - \varepsilon_\alpha] \leqslant (\omega_\alpha)_{\varepsilon_\alpha}'(v_\alpha; z_\alpha).$$

By the arbitrariness of σ, this completes the proof.

The next result is dual to Theorem 8.

THEOREM 9. *Assume that X is a reflexive Banach space and that $f_\alpha \to f_\infty$, where f_∞ is proper. Let $\bar{x}_\infty$ be such that there exist $M < +\infty$, $\mu > 0$, $\bar{\alpha}$ for which*

$$f_\alpha(x) \leqslant M, \qquad \forall x \in \bar{x}_\infty + \mu B,\ \forall \alpha \geqslant \bar{\alpha}. \tag{5.18}$$

Then for any (x_α), x_∞, (v_α), (ε_α) *satisfying*

$$\begin{aligned} \|x_\infty - \bar{x}_\infty\| < \mu, \qquad & x_\infty = s\text{-}\lim x_\alpha, \\ v_\alpha \in \partial_{\varepsilon_\alpha} f_\alpha(v_\alpha), \qquad & 0 \leqslant \varepsilon_\alpha \to 0, \end{aligned} \tag{5.19}$$

one has

$$\lim \inf_x \{f_\alpha(x) - \langle x - x_\alpha, v_\alpha\rangle\} = \lim f_\alpha(x_\alpha) = f_\infty(x_\infty) \in R, \tag{5.20}$$

and also the existence of $(\beta) \subset (\alpha)$ *and* v_∞ *such that*

$$w\text{-}\lim v_\beta = v_\infty \in \partial_0 f_\infty(x_\infty), \tag{5.21}$$

$$\lim(\langle x_\beta, v_\beta\rangle - f_\beta(x_\beta)) = \lim f_\beta^*(v_\beta) = f_\infty^*(v_\infty) \in R. \tag{5.22}$$

If $\langle x_\infty, v_\infty\rangle = \lim\langle x_\alpha, v_\alpha\rangle$, *then in* (5.22) *the* β*'s can be replaced by* α*'s. In particular, if for all* α *sufficiently large the functions* f_α *are proper convex, one has*

$$\varnothing \neq w\text{-}\overline{\lim}\, \partial_{\varepsilon_\alpha} f_\alpha(x_\alpha) \subset \partial_0 f_\infty(x_\infty), \tag{5.23}$$

$$\lim f_\alpha(x_\alpha) = f_\infty(x_\infty) \in R \tag{5.24}$$

whenever $x_\infty = s\text{-}\lim x_\alpha$, $\|x_\infty - \bar{x}_\infty\| < \mu$, $0 \leqslant \varepsilon_\alpha \to 0$.

Proof. In outline this proof is similar to that of Theorem 8, although we shall get by without using the Moreau and Rockafellar results. Comparing the situation with that of Theorem 8, we find here the roles of X and V interchanged, with the role of k^* being played here by the function h defined on X by

$$h(x) = \begin{cases} M & \text{if} \quad x \in \bar{x}_\infty + \mu B, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ and $x_\infty = s\text{-}\lim x_\alpha$, where $\|x_\infty - \bar{x}_\infty\| < \mu$. By direct argument using the form of h, one can obtain the fact that, for any $\tilde{\mu} \in (0, \mu - \|x_\infty - \bar{x}_\infty\|)$, the set

$$T = \bigcup \{\partial_\lambda h(x) \mid x \in x_\infty + \tilde{\mu} B\}$$

is norm bounded in V for all $\lambda \in [0, +\infty)$. Indeed, $v \in \partial_\lambda h(x)$ occurs if and only if $x \in \bar{x}_\infty + \mu B$ and $\mu \|v\| - \langle x - \bar{x}_\infty, v\rangle \leqslant \lambda$, which in turn implies $(\mu - \|x - \bar{x}_\infty\|)\|v\| \leqslant \lambda$. Since every $x \in x_\infty + \tilde{\mu} B$ satisfies $\|x - \bar{x}_\infty\| \leqslant \tilde{\mu} + \|x_\infty - \bar{x}_\infty\| < \mu$, this last condition yields that

$$\|v\| \leqslant \lambda(\mu - \|x - \bar{x}_\infty\|)^{-1} \leqslant \lambda(\mu - \tilde{\mu} - \|x_\infty - \bar{x}_\infty\|)^{-1} < +\infty$$

whenever $v \in \partial_\lambda h(x)$ and $x \in x_\infty + \tilde{\mu}B$. It follows that T is norm bounded, say by $r < +\infty$. Now apply the preceding to the choice

$$\lambda = 2\varepsilon + h(x_\infty) - f_\infty(x_\infty) \in (0, +\infty).$$

(One has $-\infty < f_\infty(x_\infty) \leqslant \underline{\lim}\, f_\alpha(x_\alpha) \leqslant h(x_\infty) < +\infty$, by using the properness of f_∞, Lemma 1(b), and $f_\alpha \leqslant h$ for all large α.) Note that $h(x) = h(x_\infty)$ on $x_\infty + \tilde{\mu}B$. Pick $\bar{\alpha}$ so that $x_\alpha \in x_\infty + \tilde{\mu}B$ for all $\alpha \geqslant \bar{\alpha}$, and then, using Lemma 1(b), pick $\tilde{\alpha} \geqslant \bar{\alpha}$ so that $f_\infty(x_\infty) - \varepsilon \leqslant f_\alpha(x_\alpha)$ for all $\alpha \geqslant \tilde{\alpha}$. Then for any $\alpha \geqslant \tilde{\alpha}$ and any $v \in \partial_\varepsilon f_\alpha(x_\alpha)$, the estimates

$$\begin{aligned}
h^*(v) - \langle x_\alpha, v\rangle &\leqslant f_\alpha^*(v) - \langle x_\alpha, v\rangle && (\text{by } f_\alpha \leqslant h)\\
&\leqslant \varepsilon - f_\alpha(x_\alpha) && (\text{by } v \in \partial_\varepsilon f_\alpha(x_\alpha))\\
&\leqslant \varepsilon + \varepsilon - f_\infty(x_\infty) && (\text{by } \tilde{\alpha})\\
&= \lambda - h(x_\infty) && (\text{by } \lambda)\\
&= \lambda - h(x_\alpha) && (\text{by } \bar{\alpha})
\end{aligned}$$

imply that $v \in \partial_\lambda h(x_\alpha) \subset T$ (by $\tilde{\mu}, \bar{\alpha}$). Thus, whenever $x_\infty = s\text{-}\lim x_\alpha$, $\|x_\infty - \bar{x}_\infty\| < \mu$, $\varepsilon > 0$ there exist $\lambda > 0$, $r < +\infty$, $\tilde{\alpha}$ satisfying

$$\partial_\varepsilon f_\alpha(x_\alpha) \subset \partial_\lambda h(x_\alpha) \subset \{v \in V \mid \|v\| \leqslant r\}, \qquad \forall \alpha \geqslant \tilde{\alpha}. \tag{5.25}$$

Suppose now that (v_α), (ε_α) satisfy (5.19). By (5.25), there exists $r < +\infty$ such that $\|v_\alpha\| \leqslant r$ for all α sufficiently large, so that Theorem 7 applies. From (5.4), (5.5) and (5.6) follow (5.20), (5.21) and (5.22), respectively, as well as the refinement of (5.22). The inclusion in (5.23) is always valid, by Theorem 5. For the existence part of (5.23), as well as (5.24), the idea is to invoke (5.21) and (5.20), for which we must be assured that a sequence (v_α) exists satisfying (5.19). A convenient condition guaranteeing this is for the f_α's to be proper convex for all α sufficiently large. For then, as is well known, $f_\alpha \leqslant h$ will imply $\varnothing \neq \partial_0 f_\alpha(x_\alpha) \subset \partial_{\varepsilon_\alpha} f_\alpha(x_\alpha)$ for all α sufficiently large. This completes the proof.

Let us now define approximate directional derivatives of the functions appearing in Theorem 9. Motivated by the convex case, we define these whenever $f(x)$ is finite by means of

$$f'_\varepsilon(x; z) = \inf_{\tau > 0} \left\{ \frac{f(x + \tau z) - f(x) + \varepsilon}{\tau} \right\}, \qquad \forall x, z \in X, \tag{5.26}$$

for any $\varepsilon \in [0, +\infty)$. We then have, at least when f is convex,

$$f'_0(x; z) = \lim_{\tau \downarrow 0} \frac{f(x + \tau z) - f(x)}{\tau} \tag{5.27}$$

and also

$$\inf_{\tau>0}\left\{\frac{f(x+\tau z)-f(x)+\varepsilon}{\tau}\right\}=\sup\{\langle z,v\rangle\mid v\in\partial_\varepsilon f(x)\} \tag{5.28}$$

for $\varepsilon\in[0,+\infty)$ and $z\in X$ (cf. (5.14)–(5.16)).

COROLLARY. *Under the hypotheses of Theorem* 9, *and assuming that* f_∞ *is proper convex, the approximate directional derivative functions* $(f_\alpha)'_\varepsilon(x;z)$ *defined as in* (5.26) *satisfy*

$$\overline{\lim}(f_\alpha)'_{\varepsilon_\alpha}(x_\alpha;z_\alpha)\leqslant(f_\infty)'_0(x_\infty;z_\infty) \tag{5.29}$$

whenever $x_\infty=s\text{-}\lim x_\alpha$, $\|x_\infty-\bar{x}_\infty\|<\mu$, $0\leqslant\varepsilon_\alpha\to 0$, $z_\infty=s\text{-}\lim z_\alpha$.

Proof. Like that for the Corollary to Theorem 8.

6. A SUFFICIENCY CONDITION ENSURING STRONG CONVERGENCE

Our goal in this section is to establish a sufficiency result for $\Omega_\infty(v_\infty,0)$ containing some very strong conclusions (Theorem 10) and also a dual form of it (Theorem 11).

We use the following definition for any fixed vectors $x_\infty\in X$, $v_\infty\in V$ such that $\omega_\infty(v_\infty)$ is finite. For any $\gamma>0$, consider the following property:

$$\left.\begin{array}{l}\text{there exist } \lambda>0, \mu>0 \text{ and } \tilde{\alpha} \text{ such that}\\ \lambda\mu^{-1}\leqslant\gamma \text{ and the function } k \text{ satisfies}\\ k\leqslant f_\alpha \text{ for all } \alpha\geqslant\tilde{\alpha} \text{ (excluding } \alpha=\infty),\end{array}\right\}\qquad \pi(\gamma)$$

where k is defined by

$$k(x)=\omega_\infty(v_\infty)+\langle x,v_\infty\rangle-\lambda+\mu\|x-x_\infty\|. \tag{6.1}$$

If $\pi(\gamma)$ is satisfied for every $\gamma>0$, we say that *property* π *holds* (*at* x_∞ *with respect to* v_∞). As will become apparent, property π serves as a uniform local version (at x_∞ with respect to v_∞) of the property

$$w\text{-}\overline{\lim}\operatorname{epi}f_\alpha\subset\operatorname{epi}f_\infty \tag{6.2}$$

(cf. (1.4)). As we shall elaborate presently, evidence is available to support the view that property π is satisfied generically in the reflexive Banach space setting.

THEOREM 10. *Assume that* $\operatorname{epi}f_\infty\subset s\text{-}\underline{\lim}\operatorname{epi}f_\alpha$ *and that* f_∞ *is proper.*

Let x_∞ and v_∞ be such that $\omega_\infty(v_\infty)$ is finite and property π holds. Then for every $\gamma > 0$ and $v_\infty = s\text{-}\lim v_\alpha$ there exist $\varepsilon > 0$ and $\hat{\alpha}$ such that

$$\Omega_\alpha(v_\alpha, \varepsilon) \subset x_\infty + \gamma B, \qquad \forall \alpha = \hat{\alpha},..., \infty. \tag{6.3}$$

It follows that $\Omega_\infty(v_\infty, 0) \subset \{x_\infty\}$ and, for every $(\beta) \subset (\alpha)$ and corresponding sequences (x_β), (v_β), (ε_β), that the conditions

$$x_\beta \in \Omega_\beta(v_\beta, \varepsilon_\beta), \qquad 0 \leqslant \varepsilon_\beta \to 0, \qquad v_\infty = s\text{-}\lim v_\beta \tag{6.4}$$

imply

$$x_\infty = s\text{-}\lim x_\beta. \tag{6.5}$$

If $\Omega_\infty(v_\infty, 0) = \{x_\infty\}$ and the nontriviality condition $\operatorname{dom} f_\infty \not\subset \{x_\infty\}$ *is met, then conditions* (6.4) *imply also that*

$$\lim(f_\beta(x_\beta) - \langle x_\beta, v_\beta\rangle) = \lim \omega_\beta(v_\beta) = \omega_\infty(v_\infty), \tag{6.6}$$

$$\lim \inf_x\{f_\beta(x) - \langle x - x_\beta, v_\beta\rangle\} = \lim f_\beta(x_\beta) = f_\infty(x_\infty). \tag{6.7}$$

If f_∞ is norm lower semicontinuous at x_∞, then $\Omega_\infty(v_\infty, 0) = \{x_\infty\}$ and f_∞ is norm rotund at x_∞ with respect to v_∞.

How stringent is the norm rotundity condition forced upon a proper, norm lower semicontinuous f_∞ by the hypotheses of Theorem 10? In other words, how "likely" is it that this necessary condition will be satisfiable by some x_∞ for a given parameter v_∞? Let us recall that Asplund [1, Theorem 3] (cf. also Ekeland and Lebourg [21, pp. 208–209]) has shown the following: If (i) X is a reflexive Banach space whose dual admits an equivalent Fréchet-differentiable norm, (ii) f_∞ is proper and norm lower semicontinuous on X, and (iii) $\omega_\infty(v) > -\infty$ for all v belonging to some norm neighborhood N of v_∞, then there exists a norm dense G_δ subset $G \subset N$ such that (among a number of his conclusions) one has for each $v \in G$ that f_∞ is norm rotund at some unique $x(v)$ with respect to v. More recently, Troyanski [42] has proved a result implying that the dual of every reflexive Banach space admits an equivalent Fréchet-differentiable norm. We therefore have the following: If X is a reflexive Banach space and f_∞ is proper and norm lower semicontinuous, then the previously mentioned necessary condition on f_∞ is satisfied generically by $v_\infty \in \operatorname{int}(\operatorname{dom} f_\infty^*)$. (Note: $\varnothing \neq \operatorname{int}(\operatorname{dom} f_\infty^*)$ if there exist v and $\gamma > \inf\{f_\infty^{**} - \langle \cdot, v\rangle\}$ such that $\{x \in X \mid f_\infty^{**}(x) - \langle x, v\rangle \leqslant \gamma\}$ is weakly compact.)

After proving Theorem 10 we shall present a dual version of it. In view of the norm rotundity feature just discussed, it is perhaps not surprising (cf. [2, 14]) that the dual result entails Fréchet-differentiability at a certain point.

For greater initial readability we stated Theorem 10 formally in terms of the sets $\Omega(v, \varepsilon)$. In fact, though, a sharper form will actually be proved, one in which $\Omega(v, \varepsilon)$ is replaced everywhere by the larger set $\bar{\Omega}(v, \varepsilon) = \partial_\varepsilon^* f^*(v)$. (Recall the equivalences (2.6) $\Leftrightarrow$ (2.10), (2.18) $\Leftrightarrow$ (2.22) and also (2.12), (2.16).)

THEOREM 10′. *Assume that* epi $f_\infty \subset s$-$\underline{\lim}$ epi f_α *and that the function* f_∞ *is proper. Let* x_∞ *and* v_∞ *be such that* $\omega_\infty(v_\infty)$ *is finite and property* π *holds. Then for every* $\gamma > 0$ *and* $v_\infty = s\text{-}\lim v_\alpha$ *there exist* $\varepsilon > 0$ *and* $\hat{a}$ *such that*

$$\bar{\Omega}_\alpha(v_\alpha, \varepsilon) \subset x_\infty + \gamma B, \qquad \forall \alpha = \hat{a}, \ldots, \infty. \tag{6.3′}$$

It follows that $\bar{\Omega}_\infty(v_\infty, 0) \subset \{x_\infty\}$ *and, for every* $(\beta) \subset (\alpha)$ *and corresponding sequences* (x_β), (v_β), (ε_β), *that the conditions*

$$x_\beta \in \bar{\Omega}_\beta(v_\beta, \varepsilon_\beta), \qquad 0 \leqslant \varepsilon_\beta \to 0, \qquad v_\infty = s\text{-}\lim v_\beta \tag{6.4′}$$

imply

$$x_\infty = s\text{-}\lim x_\beta. \tag{6.5′}$$

If $\Omega_\infty(v_\infty, 0) = \{x_\infty\}$ *and the nontriviality condition* dom $f_\infty \not\subset \{x_\infty\}$ *is met, then conditions* (6.4′) *imply also that*

$$\lim(f_\beta^{**}(x_\beta) - \langle x_\beta, v_\beta \rangle) = \lim \bar{\omega}_\beta(v_\beta) = \bar{\omega}_\infty(v_\infty), \tag{6.6′}$$

$$\lim \inf_x \{f_\beta^{**}(x) - \langle x - x_\beta, v_\beta \rangle\} = \lim f_\beta^{**}(x_\beta) = f_\infty^{**}(x_\infty). \tag{6.7′}$$

If f_∞ *is norm lower semicontinuous at* x_∞, *then* $\Omega_\infty(v_\infty, 0) = \bar{\Omega}_\infty(v_\infty, 0) = \{x_\infty\}$ *and* f_∞ *is norm rotund at* x_∞ *with respect to* v_∞.

Our proof of Theorems 10 and 10′ invokes the following result. Notice that assertion (6.10) applies a fortiori to the situation of Theorem 10.

PROPOSITION 3. *Assume that* epi $f_\infty \subset s$-$\underline{\lim}$ epi f_α *and that the function* f_∞ *is proper. Let* x_∞, v_∞, λ, μ *and* $\tilde{a}$ *be such that* $\omega_\infty(v_\infty)$ *is finite and the triple* $(\lambda, \mu, \tilde{a})$ *satisfies* $\pi(\gamma)$ *for* $\gamma = \lambda\mu^{-1}$. *Then for every* $\eta \in (0, 1)$ *and* $v_\infty = s\text{-}\lim v_\alpha$ *there exists* $\hat{a}$ *such that*

$$\bar{\Omega}_\alpha(v_\alpha, \eta\lambda) \subset x_\infty + \rho(\eta)B, \qquad \forall \alpha = \hat{a}, \ldots, \infty, \tag{6.8}$$

where $\rho(\eta) = (1 + 3\eta)\lambda(1 - \eta)^{-1}\mu^{-1}$ *decreases to* $\lambda\mu^{-1}$ *as* $\eta \downarrow 0$. *Thus,*

$$\bar{\Omega}_\infty(v_\infty, 0) \subset x_\infty + \lambda\mu^{-1}B. \tag{6.9}$$

In addition,

$$\varnothing \neq \Omega_\alpha(v_\alpha, 0), \qquad \forall \alpha = \hat{\alpha}, ..., \infty, \tag{6.10}$$

whenever X is the dual of another normed linear space V_0, the f_α's are weak lower semicontinuous, and the $\omega_\alpha(v_\alpha)$'s are not $-\infty$, with $\{v_\alpha \mid \alpha = 1, ..., \infty\} \subset V_0 \subset V$.*

Proof. Let $\eta \in (0, 1)$ and $v_\infty = \text{s-lim}\, v_\alpha$. Using Lemma 2, we can obtain $\hat{\alpha} \geqslant \tilde{\alpha}$ such that $\alpha > \hat{\alpha}$ implies $\omega_\alpha(v_\alpha) \leqslant \eta\lambda + \omega_\infty(v_\infty)$ as well as $v_\alpha \in v_\infty + \eta\mu B$ and $\|x_\infty\| \cdot \|v_\alpha - v_\infty\| \leqslant \eta\lambda$. Some computation shows that

$$k^*(v) = \begin{cases} \lambda - \omega_\infty(v_\infty) + \langle x_\infty, v - v_\infty \rangle & \text{if } v \in v_\infty + \mu B, \\ +\infty & \text{otherwise.} \end{cases}$$

Now consider any $\alpha \geqslant \hat{\alpha}$. Then

$$\begin{aligned} -\omega_\alpha(v_\alpha) &\geqslant -\omega_\infty(v_\infty) - \eta\lambda \\ &= (k^*(v_\alpha) - \lambda - \langle x_\infty, v_\alpha - v_\infty \rangle) - \eta\lambda \\ &\geqslant k^*(v_\alpha) - (1 + 2\eta)\lambda. \end{aligned}$$

Suppose $x \in \bar{\Omega}_\alpha(v_\alpha, \eta\lambda) = \partial^*_{\eta\lambda} f^*_\alpha(v_\alpha)$. Then for each $v \in v_\infty + \mu B$ it follows that

$$\begin{aligned} k^*(v) &\geqslant f^*_\alpha(v) \\ &\geqslant f^*_\alpha(v_\alpha) - \eta\lambda + \langle x, v - v_\alpha \rangle \\ &= -\omega_\alpha(v_\alpha) - \eta\lambda + \langle x, v - v_\alpha \rangle \\ &\geqslant k^*(v_\alpha) - (1 + 3\eta)\lambda + \langle x, v - v_\alpha \rangle, \end{aligned}$$

which yields $x \in \partial^*_\zeta k^*(v_\alpha)$ with $\zeta = (1 + 3\eta)\lambda$. But the latter condition holds if and only if

$$\langle x - x_\infty, v - v_\alpha \rangle \leqslant \zeta, \qquad \forall v \in v_\infty + \mu B.$$

Since $x_\alpha + (1 - \eta)\mu B \subset (x_\infty + \eta\mu B) + (1 - \eta)\mu B$, we can conclude that

$$\langle x - x_\infty, v - v_\alpha \rangle \leqslant \zeta, \qquad \forall v \in v_\alpha + (1 - \eta)\mu B,$$

and hence

$$\|x - x_\infty\| = \|x - x_\infty\|_{**} \leqslant \zeta(1 - \eta)^{-1}\mu^{-1} = \rho(\eta).$$

Notice that the limit index $\alpha = \infty$ is also covered by the preceding argument.

Indeed, the only missing ingredient is the fact that $k^* \geqslant f^*_\infty$. But this follows from

$$k \leqslant f_\infty, \tag{6.11}$$

which itself follows from

$$k(y) \leqslant \underline{\lim}\, k(y_\alpha) \leqslant \overline{\lim}\, f_\alpha(y_\alpha) \leqslant f_\infty(y),$$

using Lemma 1(a), the hypothesis that $k \leqslant f_\alpha$ for all α sufficiently large, and the lower semicontinuity of k. We have thus shown (6.8). Since $\bar{\Omega}_\infty(v_\infty, 0) \subset \bar{\Omega}_\infty(v_\infty, \eta\lambda)$ for every $\eta \in (0, 1)$, with $\rho(\eta)$ clearly decreasing to $\lambda\mu^{-1}$ as $\eta \downarrow 0$, it follows from (6.8) that (6.9) also holds. Finally, suppose X is the dual of some normed V_0. For each $\varepsilon > 0$ and $\alpha = 1,..., \infty$ the set $\Omega_\alpha(v_\alpha, \varepsilon)$ is weak* closed, if $v_\alpha \in V_0$ and f_α is weak* lower semicontinuous, and also nonempty if $\omega_\alpha(v_\alpha) > -\infty$. Since these sets form a decreasing nest as $\varepsilon \downarrow 0$, it follows from (6.8) and the weak* compactness of $x_\infty + \rho(\eta)B$ that, for each $\alpha = \hat{\alpha},..., \infty$, their intersection over $\varepsilon \in (0, \eta\lambda]$ is nonempty. Since this intersection is exactly $\Omega_\alpha(v_\alpha, 0)$, this establishes (6.10) and completes the proof.

Joint Proof of Theorems 10 *and* 10′. Let $\gamma > 0$ and $v_\infty = s\text{-lim}\, v_\alpha$. By property π, there exists a triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi(\gamma/2)$. Therefore, by Proposition 3, for each $\eta \in (0, 1)$ there exists $\hat{\alpha}$ such that (6.8) holds. Now pick any $\eta \in (0, 1/5]$ and then choose $\varepsilon = \eta\lambda$. Since the corresponding value $\rho(\eta)$ satisfies $\rho(\eta) \leqslant 2\lambda\mu^{-1} \leqslant \gamma$, this establishes (6.3′) (and hence also (6.3)). Since the index $\alpha = \infty$ is covered here,

$$\Omega_\infty(v_\infty, 0) \subset \bar{\Omega}_\infty(v_\infty, 0) \subset \{x_\infty\} \tag{6.12}$$

follows. Now consider any (x_β), (v_β), (ε_β) satisfying (6.4′). Observe that (6.3′) holds for all indices $\beta \geqslant \hat{\alpha}$. So, for any $\gamma > 0$ there exist $\varepsilon > 0$ and $\hat{\beta}$ such that $\bar{\Omega}_\beta(v_\beta, \varepsilon) \subset x_\infty + \gamma B$ for all $\beta \geqslant \hat{\beta}$. Pick $\bar{\beta} \geqslant \hat{\beta}$ so that $\varepsilon_\beta \leqslant \varepsilon$ for all $\beta \geqslant \bar{\beta}$. Then, for any $\beta \geqslant \bar{\beta}$,

$$x_\beta \in \bar{\Omega}_\beta(v_\beta, \varepsilon_\beta) \subset \bar{\Omega}_\beta(v_\beta, \varepsilon) \subset x_\infty + \gamma B.$$

This establishes (6.5) (for both Theorems 10 and 10′). Now assume f_∞ is norm lower semicontinuous at x_∞. Since we already have (6.12), for

$$\Omega_\infty(v_\infty, 0) = \bar{\Omega}_\infty(v_\infty, 0) = \{x_\infty\} \tag{6.13}$$

it will suffice to show $x_\infty \in \Omega_\infty(v_\infty, 0)$, that is,

$$f_\infty(x_\infty) - \langle x_\infty, v_\infty \rangle \leqslant \sigma + \omega_\infty(v_\infty), \qquad \forall \sigma > 0. \tag{6.14}$$

Let $\sigma > 0$. Using semicontinuity, pick $\gamma > 0$ so that

$$f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle - \sigma/2 \leqslant f_\infty(x) - \langle x, v_\infty\rangle, \qquad \forall x \in x_\infty + \gamma B. \quad (6.15)$$

By (6.3′), there exists $\varepsilon > 0$ such that

$$\Omega_\infty(v_\infty, \varepsilon) \subset x_\infty + \gamma B. \quad (6.16)$$

Put $\varepsilon' = \min\{\varepsilon, \sigma/2\}$. Since $\varepsilon' > 0$ and $\omega_\infty(v_\infty) > -\infty$, there exists an $x' \in \Omega_\infty(v_\infty, \varepsilon')$, that is, an x' satisfying

$$f_\infty(x') - \langle x', v_\infty\rangle \leqslant \varepsilon' + \omega_\infty(v_\infty).$$

Since $\varepsilon' \leqslant \varepsilon$, so that $\Omega_\infty(v_\infty, \varepsilon') \subset \Omega_\infty(v_\infty, \varepsilon)$, it follows from (6.16) that $x' \in x_\infty + \gamma B$ and then from (6.15) that

$$f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle \leqslant \sigma/2 + f_\infty(x') - \langle x', v_\infty\rangle.$$

Combining the last two inequalities and using $\varepsilon' \leqslant \sigma/2$ yields (6.14). Hence, (6.13) is established. Now observe that (6.3′) yields, in particular, that for each $\gamma > 0$ there exists $\varepsilon > 0$ such that $\Omega_\infty(v_\infty, \varepsilon) \subset x_\infty + \gamma B$. Since $x_\infty \in \Omega_\infty(v_\infty, 0)$ means $\omega_\infty(v_\infty) = f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle$,

$$\Omega_\infty(v_\infty, \varepsilon) = \{x \in X \mid f_\infty(x) - \langle x, v_\infty\rangle \leqslant \varepsilon + f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle\}.$$

Thus, for every $\gamma > 0$ there exists $\varepsilon > 0$ such that

$$\{x \in X \mid f_\infty(x) \leqslant \varepsilon + f_\infty(x_\infty) + \langle x - x_\infty, v_\infty\rangle\} \subset x_\infty + \gamma B.$$

This is just norm rotundity of f_∞ at x_∞ with respect to v_∞.

Finally, we tackle the assertions that (6.4) implies (6.6), (6.7), and that (6.4′) implies (6.6′), (6.7′). Analysis of the proof of Theorem 5 reveals that the strings of inequalities given there would establish that (6.4) implies both (6.6) and (6.7), provided we could justify inequalities (4.17) and (4.20) without recourse to Lemma 1(b). Thus, we shall show below that (6.4) implies

$$f_\infty(x_\infty) \leqslant \underline{\lim}\, f_\beta(x_\beta). \quad (6.17)$$

Now consider the assertion that (6.4′) implies (6.6′) and (6.7′). In outline, the argument for (6.6′) is similar to one of the two strings of inequalities used for Theorem 5:

$$\begin{aligned}
\overline{\lim}\, \bar{\omega}_\beta(v_\beta) &\leqslant \overline{\lim}(f_\beta^{**}(x_\beta) - \langle x_\beta, v_\beta\rangle) \\
&\leqslant \overline{\lim}(\varepsilon_\beta + \bar{\omega}_\beta(v_\beta)) &\quad (6.18)\\
&= \overline{\lim}\, \bar{\omega}_\beta(v_\beta) \\
&= \overline{\lim}\, \omega_\beta(v_\beta) &\quad (6.19)\\
&\leqslant \omega_\infty(v_\infty) &\quad (6.20)\\
&= \bar{\omega}_\infty(v_\infty) &\quad (6.21)\\
&\leqslant f_\infty^{**}(x_\infty) - \langle x_\infty, v_\infty\rangle \\
&\leqslant \underline{\lim}(f_\beta^{**}(x_\beta) - \langle x_\beta, v_\beta\rangle) &\quad (6.22)\\
&\leqslant \underline{\lim}(\varepsilon_\beta + \bar{\omega}_\beta(v_\beta)) &\quad (6.23)\\
&= \underline{\lim}\, \bar{\omega}_\beta(v_\beta).
\end{aligned}$$

We need to justify the numbered steps. Steps (6.18) and (6.23) follow from (6.4′). Steps (6.19) and (6.21) follow from (2.16). Step (6.20) follows from Lemma 2. This leaves only step (6.22), which will follow once we show

$$\langle x_\infty, v_\infty\rangle = \lim\langle x_\beta, v_\beta\rangle, \tag{6.24}$$

$$f_\infty^{**}(x_\infty) \leqslant \underline{\lim}\, f_\beta^{**}(x_\beta). \tag{6.25}$$

Recall that we previously proved (6.4′) implies (6.5). Together with $v_\infty = s\text{-}\lim v_\beta$, this yields (6.24). In order to complete the proof of (6.6′), it will therefore suffice to show (6.4′) implies (6.25). We defer this for a moment and consider the argument leading to (6.7′). Consider the following string of inequalities, also patterned after the proof of Theorem 5:

$$\begin{aligned}
\overline{\lim}(\bar{\omega}_\beta(v_\beta) + \langle x_\beta, v_\beta\rangle) &\leqslant \overline{\lim}\, f_\beta^{**}(x_\beta) \\
&\leqslant \overline{\lim}\langle x_\beta, v_\beta\rangle + \overline{\lim}(f_\beta^{**}(x_\beta) - \langle x_\beta, v_\beta\rangle) \\
&\leqslant \langle x_\infty, v_\infty\rangle + (f_\infty^{**}(x_\infty) - \langle x_\infty, v_\infty\rangle) &\quad (6.26)\\
&= f_\infty^{**}(x_\infty) \\
&\leqslant \underline{\lim}\, f_\beta^{**}(x_\beta) &\quad (6.27)\\
&\leqslant \underline{\lim}(\varepsilon_\beta + \bar{\omega}_\beta(v_\beta) + \langle x_\beta, v_\beta\rangle) \\
&= \underline{\lim}(\bar{\omega}_\beta(v_\beta) + \langle x_\beta, v_\beta\rangle).
\end{aligned}$$

Step (6.26) here would follow from (6.24) and (6.6′), while step (6.27) is precisely (6.25). It follows that in order to complete the proof that (6.4′) implies both (6.6′) and (6.7′),it suffices to show (6.4′) implies (6.25).Recall now the only unfinished part of the argument showing (6.4) implies both (6.6) and (6.7) is the assertion that (6.4) implies (6.17).

We shall now give a joint proof that (6.4) implies (6.17) and that (6.4′) implies (6.25). This will complete the joint proof of Theorems 10 and 10′. There are two cases to consider, described in terms of the quantity

$$m = \inf\{\lambda > 0 \mid \exists\gamma > 0\ \exists\mu > 0\ \exists\tilde{\alpha},\ (\lambda, \mu, \tilde{\alpha}) \text{ satisfies } \pi(\gamma)\}. \tag{6.28}$$

Case I: $m = 0$. (Here, we deduce both (6.17) and (6.25).) For any $\varepsilon > 0$ there exist $\gamma > 0$ and a corresponding triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi(\gamma)$ and also $2\lambda \leqslant \varepsilon$. We have $k \leqslant f_\alpha$ for all $\alpha = \hat{\alpha},..., \infty$ (the inclusion of $\alpha = \infty$ is justified by (6.11)). Since $k \geqslant \text{const} + \langle\cdot, v_\infty\rangle$, it follows that there exists $\tilde{\beta} \geqslant \tilde{\alpha}$ such that

$$f_\beta \geqslant f_\beta^{**} \geqslant k, \qquad \forall\beta = \tilde{\beta},..., \infty. \tag{6.29}$$

Now by (6.5), which follows from either (6.4) or (6.4′), pick $\bar{\beta} \geqslant \tilde{\beta}$ so that

$$2\,\|x_\beta - x_\infty\| \cdot \|v_\infty\| \leqslant \varepsilon, \qquad \forall\beta \geqslant \bar{\beta}. \tag{6.30}$$

Consider any $\beta \geqslant \bar{\beta}$. Using (6.29), (6.1), $\omega_\infty(v_\infty) = f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle$ (from $x_\infty \in \Omega_\infty(v_\infty, 0)$) and (6.30), we obtain

$$\begin{aligned} f_\beta(x_\beta) &\geqslant f_\beta^{**}(x_\beta) \\ &\geqslant k(x_\beta) \\ &\geqslant f_\infty(x_\infty) + \langle x_\beta - x_\infty, v_\infty\rangle - \lambda + 0 \\ &\geqslant f_\infty(x_\infty) - \varepsilon \\ &\geqslant f_\infty^{**}(x_\infty) - \varepsilon. \end{aligned}$$

By the arbitrariness of ε, this establishes both (6.17) and (6.25).

Case II: $m > 0$. (Here, we deduce $\operatorname{dom} f_\infty \subset \{x_\infty\}$.) Consider any $\bar{x} \neq x_\infty$, and suppose $f_\infty(\bar{x}) < +\infty$. Pick any γ satisfying $0 < \gamma < m(\delta + m)^{-1}$ $\|\bar{x} - x_\infty\|$, where $\delta = f_\infty(\bar{x}) - \langle \bar{x}, v_\infty\rangle - \omega_\infty(v_\infty) \geqslant 0$. Then

$$\begin{aligned} -\lambda + \mu\,\|\bar{x} - x_\infty\| &\geqslant -\lambda + \lambda\gamma^{-1}\,\|\bar{x} - x_\infty\| \\ &\geqslant m(-1 + \gamma^{-1}\,\|\bar{x} - x_\infty\|) \\ &> -m + \delta + m, \end{aligned}$$

so that

$$\begin{aligned} k(\bar{x}) &= \omega_\infty(v_\infty) + \langle \bar{x}, v_\infty \rangle - \lambda + \mu \|\bar{x} - x_\infty\| \\ &> \omega_\infty(v_\infty) + \langle \bar{x}, v_\infty \rangle + \delta \\ &= f_\infty(\bar{x}). \end{aligned}$$

Since this contradicts $f_\infty \geqslant k$ (recall (6.11)), we must have $f_\infty(\bar{x}) = +\infty$. This establishes that $\operatorname{dom} f_\infty \subset \{x_\infty\}$ in Case II. By the nontriviality assumption, it follows that Case I must occur. This completes the proof of Theorems 10 and 10′.

The result dual to Theorems 10 and 10′ will use the following definition for any fixed vectors $x_\infty \in X$, $v_\infty \in V$ such that $f_\infty(x_\infty)$ is finite. For any $\gamma > 0$, consider the following property:

$$\left.\begin{array}{l} \text{there exist } \lambda > 0, \mu > 0 \text{ and } \tilde{\alpha} \text{ such that} \\ \lambda\mu^{-1} \leqslant \gamma \text{ and the function } h \text{ satisfies} \\ f_\alpha \leqslant h \text{ for all } \alpha \geqslant \tilde{\alpha} \text{ (excluding } \alpha = \infty), \end{array}\right\} \qquad \pi^*(\gamma)$$

where h is defined by

$$h(x) = \begin{cases} \lambda + f_\infty(x_\infty) + \langle x - x_\infty, v_\infty \rangle & \text{if } \; x \in x_\infty + \mu B, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.31}$$

If $\pi^*(\gamma)$ is satisfied for every $\gamma > 0$, we say that *property* π^* *holds* (*at* x_∞ *with respect to* v_∞). As will become evident, property π^* serves as a uniform local version (at x_∞ with respect to v_∞) of the property

$$\operatorname{epi} f_\infty \subset s\text{-}\underline{\lim} \operatorname{epi} f_\alpha$$

(cf. (1.4)).

THEOREM 11. *Assume that* $w\text{-}\overline{\lim} \operatorname{epi} f_\alpha \subset \operatorname{epi} f_\infty$ *and that the function* f_∞ *is proper. Let* x_∞ *and* v_∞ *be such that* $f_\infty(x_\infty)$ *is finite and property* π^* *holds. Then for every* $\gamma > 0$ *and* $x_\infty = s\text{-}\lim x_\alpha$ *there exist* $\varepsilon > 0$ *and* $\hat{\alpha}$ *such that*

$$\partial_\varepsilon f_\alpha(x_\alpha) \subset v_\infty + \gamma B, \qquad \forall \alpha = \hat{\alpha}, \ldots, \infty. \tag{6.32}$$

It follows that $\partial_0 f_\infty(x_\infty) \subset \{v_\infty\}$ *and, for every* $(\beta) \subset (\alpha)$ *and corresponding sequences* (x_β), (v_β), (ε_β), *that the conditions*

$$v_\beta \in \partial_{\varepsilon_\beta} f_\beta(x_\beta), \qquad 0 \leqslant \varepsilon_\beta \to 0, \qquad x_\infty = s\text{-}\lim x_\beta \tag{6.33}$$

imply

$$v_\infty = s\text{-}\lim v_\beta. \tag{6.34}$$

Conditions (6.33) *imply also that*

$$f_\infty(x_\infty) = \lim f_\beta(x_\beta) = \lim \inf_x \{f_\beta(x) - \langle x - x_\beta, v_\beta\rangle\}, \tag{6.35}$$

$$f^*_\infty(v_\infty) = \lim f^*_\beta(v_\beta) = \lim(\langle x_\beta, v_\beta\rangle - f_\beta(x_\beta)), \tag{6.36}$$

provided $\partial_0 f_\infty(x_\infty) = \{v_\infty\}$ *and* f_∞ *satisfies the following nontriviality condition: the largest weakly lower semicontinuous convex minorant of* $f_\infty - \langle \cdot, v_\infty\rangle$ *is not constantly equal to* $\omega_\infty(v_\infty) = \inf_x\{f_\infty(x) - \langle x, v_\infty\rangle\}$. *If* f_∞ *is convex, then* $\partial_0 f_\infty(x_\infty) = \{v_\infty\}$ *and* f_∞ *is Fréchet-differentiable at* x_∞ *with* $\nabla f_\infty(x_\infty) = v_\infty$. *If* f_∞ *is convex and weakly lower semicontinuous, it satisfies the nontriviality condition unless* $f_\infty(x) = \langle x, v_\infty\rangle + \omega_\infty(v_\infty)$.

How stringent is the Fréchet-differentiability condition forced upon a proper convex f_∞ by the hypotheses of Theorem 11? Combining the previously cited result of Troyanski [42] with another theorem of Asplund [1, Theorem 1], we have the following: If X is a reflexive Banach space and f_∞ is proper convex, then f_∞ is Fréchet-differentiable on a dense G_δ subset of int(dom f_∞).

Our proof of Theorem 11 relies on the following result, which is dual to Proposition 3.

PROPOSITION 4. *Assume that* $w\text{-}\overline{\lim}\operatorname{epi} f_\infty \subset \operatorname{epi} f_\infty$ *and that the function* f_∞ *is proper. Let* x_∞, v_∞, $\lambda > 0$, $\mu > 0$ *and* $\tilde{\alpha}$ *be such that* $f_\infty(x_\infty)$ *is finite and the triple* $(\lambda, \mu, \tilde{\alpha})$ *satisfies property* $\pi^*(\gamma)$ *for* $\gamma = \lambda\mu^{-1}$. *Then for every* $\eta \in (0, 1)$ *and* $x_\infty = s\text{-}\lim x_\alpha$ *there exists* $\hat{\alpha}$ *such that*

$$\partial_{\eta\lambda} f_\alpha(x_\alpha) \subset v_\infty + \rho(\eta)B, \qquad \forall \alpha = \hat{\alpha}, \ldots, \infty, \tag{6.37}$$

where $\rho(\eta) = (1 + 3\eta)\lambda(1 - \eta)^{-1}\mu^{-1}$ *decreases to* $\lambda\mu^{-1}$ *as* $\eta \downarrow 0$. *Thus,*

$$\partial_0 f_\infty(x_\infty) \subset v_\infty + \lambda\mu^{-1}B. \tag{6.38}$$

In addition,

$$\varnothing \neq \partial_0 f_\alpha(x_\alpha), \qquad \forall \alpha = \hat{\alpha}, \ldots, \infty, \tag{6.39}$$

provided the functions $f_1, f_2, \ldots, f_\infty$ *are proper convex.*

Proof. Let $\eta \in (0, 1)$ and $x_\infty = s\text{-}\lim x_\alpha$. Using Lemma 1(b), we can obtain $\hat{\alpha} \geqslant \tilde{\alpha}$ such that $\alpha \geqslant \hat{\alpha}$ implies $f_\infty(x_\infty) - \eta\lambda \leqslant f_\alpha(x_\alpha)$ as well as $x_\alpha \in x_\infty + \eta\mu B$ and $\|x_\alpha - x_\infty\| \cdot \|v_\infty\| \leqslant \eta\lambda$. Now consider any $\alpha \geqslant \hat{\alpha}$. Then

$$\begin{aligned} f_\alpha(x_\alpha) &\geqslant f_\infty(x_\infty) - \eta\lambda \\ &= (h(x_\alpha) - \lambda - \langle x_\alpha - x_\infty, v_\infty \rangle) - \eta\lambda \\ &\geqslant h(x_\alpha) - (1 + 2\eta)\lambda. \end{aligned}$$

Suppose $v \in \partial_{\eta\lambda} f_\alpha(x_\alpha)$. Then for each $x \in x_\infty + \mu B$ it follows that

$$\begin{aligned} h(x) &\geqslant f_\alpha(x) \\ &\geqslant f_\alpha(x_\alpha) - \eta\lambda + \langle x - x_\alpha, v \rangle \\ &\geqslant h(x_\alpha) - (1 + 3\eta)\lambda + \langle x - x_\alpha, v \rangle, \end{aligned}$$

which yields $v \in \partial_\zeta h(x_\alpha)$ with $\zeta = (1 + 3\eta)\lambda$. But the latter condition holds if and only if

$$\langle x - x_\alpha, v - v_\infty \rangle \leqslant \zeta, \qquad \forall x \in x_\infty + \mu B.$$

Since $x_\alpha + (1 - \eta)\mu B \subset (x_\infty + \eta\mu B) + (1 - \eta)\mu B$, we can conclude that

$$\langle x - x_\alpha, v - v_\infty \rangle \leqslant \zeta, \qquad \forall x \in x_\alpha + (1 - \eta)\mu B,$$

and hence $\|v - v_\infty\| \leqslant \zeta(1 - \eta)^{-1}\mu^{-1} = \rho(\eta)$. Notice that the limit index $\alpha = \infty$ is also covered by the preceding argument. Indeed, the only missing ingredient is the fact that

$$f_\infty \leqslant h, \tag{6.40}$$

which follows from Lemma 1(b) and the hypothesis that $f_\alpha \leqslant h$ for all α sufficiently large. We have therefore shown (6.37). Since $\partial_0 f_\infty(x_\infty) \subset \partial_{\eta\lambda} f_\infty(x_\infty)$ for every $\eta \in (0, 1)$, with $\rho(\eta)$ decreasing to $\lambda\mu^{-1}$ as $\eta \downarrow 0$, (6.38) follows from (6.37). Finally, let $f_1, f_2, \ldots, f_\infty$ all be proper convex. Then for any $\alpha = \hat{\alpha}, \ldots, \infty$ we have that f_α is convex, never $-\infty$, and bounded above on the neighborhood $U_\alpha = x_\alpha + (1 - \eta)\mu B$ of x_α. The bound follows from $U_\alpha \subset x_\infty + \mu B$ and the fact that on $x_\infty + \mu B$ we have $f_\alpha(x) \leqslant h(x) \leqslant \lambda + f_\infty(x_\infty) + \mu\|v_\infty\|$. It is well known that this information is sufficient to imply $\varnothing \neq \partial_0 f_\alpha(x_\alpha)$. This completes the proof.

Proof of Theorem 11. Let $\gamma > 0$ and $x_\infty = s\text{-}\lim x_\alpha$. By property π^*, there exists a triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi^*(\gamma/2)$. Therefore by Proposition 4, for each $\eta \in (0, 1)$ there exists $\hat{\alpha}$ such that (6.37) holds. Now pick any $\eta \in (0, 1/5]$ and then choose $\varepsilon = \eta\lambda$. Since the corresponding $\rho(\eta)$

satisfies $\rho(\eta) \leqslant 2\lambda\mu^{-1} \leqslant \gamma$, this establishes (6.32). Since $\alpha = \infty$ is covered here,

$$\partial_0 f_\infty(x_\infty) \subset \{v_\infty\} \tag{6.41}$$

follows. Now consider any (x_β), (v_β), (ε_β) satisfying (6.33). Observe that (6.32) holds for all indices $\beta \geqslant \hat{\alpha}$. So, for any $\gamma > 0$ there exist $\varepsilon > 0$ and $\hat{\beta}$ such that $\partial_\varepsilon f_\beta(x_\beta) \subset v_\infty + \gamma B$ for all $\beta \geqslant \hat{\beta}$. Pick $\bar{\beta} \geqslant \hat{\beta}$ so that $\varepsilon_\beta \leqslant \varepsilon$ for all $\beta \geqslant \bar{\beta}$. Then, for any $\beta \geqslant \bar{\beta}$,

$$v_\beta \in \partial_{\varepsilon_\beta} f_\beta(x_\beta) \subset \partial_\varepsilon f_\beta(x_\beta) \subset v_\infty + \gamma B.$$

This establishes (6.34). Now assume f_∞ is convex. Since $f_\infty \leqslant h$ (see (6.40)), f_∞ is bounded above by $\lambda + f_\infty(x_\infty) + \mu \|v_\infty\|$ on $x_\infty + \mu B$. Since f_∞ is never $-\infty$, this is enough to imply $\varnothing \neq \partial_0 f_\infty(x_\infty)$. In view of (6.41), this yields $\partial_0 f_\infty(x_\infty) = \{v_\infty\}$. Now let us show f_∞ is Fréchet-differentiable at x_∞ with $\nabla f_\infty(x_\infty) = v_\infty$, that is,

$$\limsup_{\tau \downarrow 0} \sup_{z \in B} \left| \frac{f_\infty(x_\infty + \tau z) - f_\infty(x_\infty)}{\tau} - \langle z, v_\infty \rangle \right| = 0. \tag{6.42}$$

Let $\varepsilon > 0$. By property π^*, there exists a triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi^*(\varepsilon)$. It follows that, for any $z \in B$ and $\tau \in (0, \mu]$,

$$\begin{aligned}
0 &\leqslant \tau^{-1}(f_\infty(x_\infty + \tau z) - f_\infty(x_\infty)) - \langle z, v_\infty \rangle && (6.43)\\
&\leqslant \mu^{-1}(f_\infty(x_\infty + \mu z) - f_\infty(x_\infty)) - \langle z, v_\infty \rangle && (6.44)\\
&\leqslant \lambda\mu^{-1} && (6.45)\\
&\leqslant \varepsilon.
\end{aligned}$$

Here, (6.43) follows from $v_\infty \in \partial_0 f_\infty(x_\infty)$, (6.45) follows from $f_\infty(x_\infty + \mu z) \leqslant h(x_\infty + \mu z)$ (recall (6.40)) and (6.44) follows from the fact (due to the convexity of f_∞) that the difference quotients appearing are nondecreasing in the real parameter. It follows that

$$\sup_{z \in B} \left| \frac{f_\infty(x_\infty + \tau z) - f_\infty(x_\infty)}{\tau} - \langle z, v_\infty \rangle \right| \leqslant \varepsilon, \qquad \tau \in (0, \mu].$$

This establishes (6.42).

Finally, let conditions (6.33) be satisfied once more. For assertions (6.35), (6.36) we consider the following string of inequalities:

$$
\begin{aligned}
\overline{\lim}(\langle x_\beta, v_\beta\rangle - f_\beta^*(v_\beta)) &\leqslant \overline{\lim}\, f_\beta(x_\beta) \\
&\leqslant f_\infty(x_\infty) \qquad (6.46)\\
&\leqslant \underline{\lim}\, f_\beta(x_\beta) \qquad (6.47)\\
&\leqslant \underline{\lim}(\varepsilon_\beta + \langle x_\beta, v_\beta\rangle - f_\beta^*(v_\beta))\\
&= \underline{\lim}(\langle x_\beta, v_\beta\rangle - f_\beta^*(v_\beta)).
\end{aligned}
$$

Inequality (6.47) follows from Lemma 1(b). If we can establish (6.46), it will follow that

$$\lim(\langle x_\beta, v_\beta\rangle - f_\beta^*(v_\beta)) = \lim f_\beta(x_\beta) = f_\infty(x_\infty), \qquad (6.48)$$

which is just (6.35). Using (6.48) together with (6.34) and $v_\infty \in \partial_0 f_\infty(x_\infty)$, one can obtain the estimate

$$
\begin{aligned}
\underline{\lim}(\langle x_\beta, v_\beta\rangle - f_\beta(x_\beta)) &\geqslant \underline{\lim}(f_\beta^*(v_\beta) - \varepsilon_\beta)\\
&= \underline{\lim}\, f_\beta^*(v_\beta)\\
&\geqslant \underline{\lim}\langle x_\beta, v_\beta\rangle + \underline{\lim}(f_\beta^*(v_\beta) - \langle x_\beta, v_\beta\rangle)\\
&\geqslant \langle x_\infty, v_\infty\rangle - f_\infty(x_\infty)\\
&= f_\infty^*(v_\infty),
\end{aligned}
$$

as well as the analogous estimate with inequalities reversed and limits superior (and the ε_β's suppressed). This would establish (6.36).

The proof of Theorem 11 will therefore be finished once we establish (6.46). For this, consider the quantity

$$m^* = \inf\{\lambda > 0 \mid \exists \gamma > 0\ \exists \mu > 0\ \exists \tilde{\alpha},\ (\lambda, \mu, \tilde{\alpha}) \text{ satisfies } \pi^*(\gamma)\}. \qquad (6.49)$$

Case I: $m^* = 0$. (Here, we deduce (6.46).) For any $\varepsilon > 0$, there exist $\gamma > 0$ and a corresponding triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi^*(\gamma)$ and also $\lambda \leqslant \varepsilon$. Now pick $\bar{\beta} \geqslant \tilde{\alpha}$ so that $\|x_\beta - x_\infty\| \leqslant \mu$ for all $\beta \geqslant \bar{\beta}$. It follows that

$$
\begin{aligned}
\overline{\lim}\, f_\beta(x_\beta) &\leqslant \overline{\lim}(f_\beta(x_\beta) - \langle x_\beta, v_\beta\rangle) + \overline{\lim}\langle x_\beta, v_\beta\rangle\\
&\leqslant \overline{\lim}(h(x_\beta) - \langle x_\beta, v_\beta\rangle) + \langle x_\infty, v_\infty\rangle \qquad (6.50)\\
&= \overline{\lim}(\lambda + f_\infty(x_\infty) - \langle x_\infty, v_\infty\rangle) + \langle x_\infty, v_\infty\rangle \qquad (6.51)\\
&\leqslant \varepsilon + f_\infty(x_\infty).
\end{aligned}
$$

For (6.50) and (6.51), use (6.34) and consider that eventually $\beta \geqslant \bar{\beta} \geqslant \tilde{\alpha}$. By the arbitrariness of ε, this establishes (6.46).

Case II: $m^* > 0$. (Here, we deduce that the largest weakly lower semicontinuous convex minorant of $f_\infty - \langle \cdot, v_\infty \rangle$ is constantly equal to the value $\inf\{f_\infty - \langle \cdot, v_\infty \rangle\} = \omega_\infty(v_\infty)$.) Consider any $\gamma > 0$ and corresponding triple $(\lambda, \mu, \tilde{\alpha})$ satisfying property $\pi^*(\gamma)$. Since $f_\infty \leqslant h$ (recall (6.40)), we therefore have

$$f_\infty(x) - \langle x, v_\infty \rangle \leqslant h_0(x) = \lambda + f_\infty(x_\infty) - \langle x_\infty, v_\infty \rangle + \psi_{x_\infty + \mu B}(x)$$

(where ψ_C equals 0 on C and $+\infty$ off C), which implies

$$(f_\infty - \langle \cdot, v_\infty \rangle)^* (v) \geqslant h_0^*(v) = -\lambda - f_\infty(x_\infty) + \langle x_\infty, v + v_\infty \rangle + \mu \|v\|$$

and hence

$$(f_\infty - \langle \cdot, v_\infty \rangle)^* + f_\infty(x_\infty) - \langle x_\infty, \cdot + v_\infty \rangle \geqslant -\lambda + \mu \|\cdot\|.$$

Using $\lambda \geqslant m^* > 0$, we obtain

$$\begin{aligned} &1/m^* \{(f_\infty - \langle \cdot, v_\infty \rangle)^* + f_\infty(x_\infty) - \langle x_\infty, \cdot + v_\infty \rangle\} \\ &\quad \geqslant \lambda^{-1}\mu \|\cdot\| - 1 \geqslant \gamma^{-1} \|\cdot\| - 1. \end{aligned}$$

By property π^*, in this estimate γ can be taken arbitrarily near zero. It follows that

$$1/m^* \{(f_\infty - \langle \cdot, v_\infty \rangle)^* (v) + f_\infty(x_\infty) - \langle x_\infty, v + v_\infty \rangle\} = +\infty, \qquad \forall 0 \neq v \in V,$$

and hence

$$(f_\infty - \langle \cdot, v_\infty \rangle)^* (v) = +\infty, \qquad \forall 0 \neq v \in V.$$

Since $(f_\infty - \langle \cdot, v_\infty \rangle)^* (0) = -\omega_\infty(v_\infty) \in R$, where finiteness follows from $v_\infty \in \partial_0 f_\infty(x_\infty)$ and the properness of f_∞, we obtain

$$(f_\infty - \langle \cdot, v_\infty \rangle)^* (v) = -\omega_\infty(v_\infty) + \psi_{\{0\}}(v), \qquad \forall v \in V.$$

Taking conjugates yields

$$(f_\infty - \langle \cdot, v_\infty \rangle)^{**} (x) = \omega_\infty(v_\infty), \qquad \forall x \in X.$$

Therefore, since the nontriviality condition on f_∞ rules out precisely this situation, Case I must occur, and thus (6.46) holds. When f_∞ is convex and weakly lower semicontinuous, then (since f_∞ is also proper by assumption) one has $(f_\infty - \langle \cdot, v_\infty \rangle)^{**} = f_\infty - \langle \cdot, v_\infty \rangle$, and so the nontriviality condition can fail only if $f_\infty(x) - \langle x, v_\infty \rangle = \omega_\infty(v_\infty)$ for every $x \in X$. This concludes the proof.

Refinements of these results hold when X is the dual of some other normed linear space V_0. On X one systematically substitutes the weak* topology induced by V_0 in place of the weak topology, and one restricts the parameters v to lie in $V_0 \subset V$. The modified proofs exploit the refinement in Lemma 2, the remarks following (2.22), and also, for example, the weak* lower semicontinuity of $\|x\|$.

7. The Finite-Dimensional Case

Due to the availability of neighborhoods having finitely many extreme points, significant refinements of the previous results are possible in finite-dimensional spaces. The refinements are based on the presence of certain uniformities which, in turn, derive from two finite-dimensional results for convex functions: Lemma 4 below, and a result of Rockafellar asserting that pointwise convergence implies uniform convergence on compacta [40, Theorem 10.8].

Various forerunners of the following result have been observed by several researchers, including Robert [37, Proposition 4.10] and Salinetti and Wets [41, Corollary 3B].

LEMMA 4. *Assume that X is finite-dimensional and that* $\operatorname{epi} f_\infty \subset \underline{\lim} \operatorname{epi} f_\alpha$, *where each function $f_1, f_2, \ldots, f_\infty$ is convex. Let $\bar{x}_\infty \in \operatorname{int}(\operatorname{dom} f_\infty)$. Then for every $\lambda > 0$ there exist $\mu > 0$ and $\tilde{\alpha}$ such that the function h defined on X by*

$$h(x) = \begin{cases} \lambda + f_\infty(x_\infty) & \text{if } \; x \in \bar{x}_\infty + \mu B, \\ +\infty & \text{otherwise}, \end{cases}$$

satisfies

$$f_\alpha \leqslant h, \qquad \forall \alpha = \tilde{\alpha}, \ldots, \infty. \tag{7.1}$$

In particular, there exist $M < +\infty$, $\mu > 0$, $\bar{\alpha}$ such that

$$f_\alpha(x) \leqslant M, \qquad \forall x \in \bar{x}_\infty + \mu B, \quad \forall \alpha = \bar{\alpha}, \ldots, \infty, \tag{7.2}$$

and

$$\overline{\lim}\, f_\alpha(x_\alpha) \leqslant f_\infty(\bar{x}_\infty) \qquad \text{whenever} \quad \bar{x}_\infty = \lim x_\alpha. \tag{7.3}$$

These assertions also hold if $f_1, f_2, \ldots$ are merely quasi-convex and f_∞ is merely upper semicontinuous at $\bar{x}_\infty$.

Proof. Clearly, (7.2) and (7.3) will follow from (7.1). Now let $\lambda > 0$, and without loss of generality suppose $\bar{x}_\infty = 0$. For (7.1), it suffices to exhibit $\mu > 0$ and $\tilde{\alpha}$ such that

$$f_\alpha(x) \leqslant \lambda + f_\infty(0), \qquad \forall x \in \mu B, \quad \forall \alpha = \tilde{\alpha}, \ldots, \infty. \tag{7.4}$$

Since f_∞ is upper semicontinuous at $\bar{x}_\infty$ (e.g., [40. Theorem 10.1]), there exists $\gamma > 0$ such that

$$f_\infty(x) \leqslant \lambda/2 + f_\infty(0), \qquad \forall x \in 2\gamma B. \tag{7.5}$$

By finite-dimensionality, there is a (full dimensional) standard simplex P centered at 0 and contained in γB. Suppose its vertices are labeled x^j for $j = 0, 1, \ldots, m$. By Lemma 1(a), for each j there exists (x^j_α) such that

$$x^j = \lim x^j_\alpha, \qquad \overline{\lim}\, f_\alpha(x^j_\alpha) \leqslant f_\infty(x^j). \tag{7.6}$$

Now observe there exists $\mu > 0$ sufficiently small that

$$x^j + \mu B \subset 2\gamma B, \qquad \forall j, \tag{7.7}$$

and also

$$\mu B \subset \operatorname*{conv}_j \{\hat{x}^j\} \qquad \text{whenever} \qquad \hat{x}^j \in x^j + \mu B, \qquad \forall j. \tag{7.8}$$

(Indeed, one can suppose B corresponds to the Euclidean norm and then take any $\mu \in (0, \hat{\mu}]$, where $\hat{\mu}$ is half the distance between the origin and the $(m-1)$-dimensional faces of P.) By (7.6), for each j there exists α^j such that

$$x^j_\alpha \in x^j + \mu B \qquad \text{and} \qquad f_\alpha(x^j_\alpha) \leqslant \lambda/2 + f_\infty(x^j), \qquad \forall \alpha \geqslant \alpha^j. \tag{7.9}$$

Put $\tilde{\alpha} = \max\{\alpha^j \mid j = 0, 1, \ldots, m\}$, and consider any $\alpha \geqslant \tilde{\alpha}$. We have $x^j_\alpha \in x^j + \mu B$ for every j (by (7.9)), hence $\mu B \subset \operatorname{conv}_j\{x^j_\alpha\}$ (by (7.8)). Then for any $x \in \mu B$ we obtain

$$\begin{aligned} f_\alpha(x) &\leqslant \max_j \{f_\alpha(x^j_\alpha)\} && \text{(quasi-convexity of } f_\alpha\text{)}, \\ &\leqslant \lambda/2 + \max_j \{f_\infty(x^j)\} && \text{(by (7.9))}, \\ &\leqslant \lambda + f_\infty(0) && \text{(by (7.7), (7.5))}. \end{aligned}$$

This completes the proof.

We present two theorems summarizing the main finite-dimensional refinements of the earlier results for $P_1(v_1), P_2(v_2), \ldots, P_\infty(v_\infty)$. The first deals with the nonconvex case.

THEOREM 12. *Assume that X is finite-dimensional and that $f_\alpha \to f_\infty$, where f_∞ is proper. Let $\bar{v}_\infty$ be such that there exists a proper convex lower semicontinuous function k on X such that $k \leqslant f_\infty$ and $\{x \in X \mid k(x) - \langle x, \bar{v}_\infty \rangle \leqslant \xi\}$ is nonempty and bounded for some $\xi \in R$. Assume also that there exist $r < +\infty$ and $\bar{\alpha}$ such that*

$$f_\alpha(x) = +\infty \qquad \text{whenever} \quad \|x\| > r, \qquad \forall \alpha \geqslant \bar{\alpha}. \tag{7.10}$$

Then there exists $\mu > 0$ such that each of the following properties holds for $C = \{v \in V \mid \|v - \bar{v}_\infty\| < \mu\}$.

(a) *Whenever $v_\alpha \to v_\infty \in C$ and $0 < \varepsilon_\alpha \to 0$, one has*

$$\varnothing \neq \overline{\lim}\, \Omega_\alpha(v_\alpha, \varepsilon_\alpha) \subset \Omega_\infty(v_\infty, 0), \tag{7.11}$$

$$\lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R. \tag{7.12}$$

(b) *More generally, whenever (x_α), (v_α), v_∞, (ε_α) satisfy*

$$x_\alpha \in \Omega_\alpha(v_\alpha, \varepsilon_\alpha), \qquad v_\alpha \to v_\infty \in C, \qquad 0 \leqslant \varepsilon_\alpha \to 0, \tag{7.13}$$

one has

$$\lim(f_\alpha(x_\alpha) - \langle x_\alpha, v_\alpha \rangle) = \lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R \tag{7.14}$$

and the existence of $(\beta) \subset (\alpha)$ and x_∞ such that

$$x_\beta \to x_\infty \in \Omega_\infty(v_\infty, 0). \tag{7.15}$$

(c) *The functions ω_α converge pointwise to ω_∞ everywhere on C, and this convergence is uniform on $\{v \in V \mid \|v - \bar{v}_\infty\| \leqslant \bar{\mu}\}$ for every $\bar{\mu} \in (0, \mu)$.*

(d) *Whenever $v_\alpha \to v_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ and $z_\alpha \to z_\infty$, the approximate directional derivative functions $\omega'_\varepsilon(v; z)$ (defined in (5.14)) satisfy*

$$(\omega_\infty)'_0\,(v_\infty; z_\infty) \leqslant \underline{\lim}(\omega_\alpha)'_{\varepsilon_\alpha}\,(v_\alpha; z_\alpha). \tag{7.16}$$

(e) *Whenever $v_\alpha \to v_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ and $\gamma > 0$, there exists $\hat{\alpha}$ such that*

$$\Omega_\alpha(v_\alpha, \varepsilon_\alpha) \subset \Omega_\infty(v_\infty, 0) + \gamma B, \qquad \forall \alpha \geqslant \hat{\alpha}, \tag{7.17}$$

$$\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha) \subset \bar{\Omega}_\infty(v_\infty, 0) + \gamma B, \qquad \forall \alpha \geqslant \hat{\alpha}. \tag{7.18}$$

(The sets $\bar{\Omega}(v, \varepsilon)$ are defined in (2.13)–(2.15).)

Proof. Parts (a), (b) and (d) will follow basically from Lemma 3, Lemma 4 and Theorem 8 and its Corollary. Then, parts (c) and (e) will follow from (a) and (d), with the aid of [40, Theorem 10.8]. We begin by observing that $f_\alpha \to f_\infty$ and (7.10) imply $\operatorname{epi} f_\infty^* \subset \underline{\lim} \operatorname{epi} f_\alpha^*$, by Lemma 3 together with (2.2) and Lemma 1(a). (Note: Lemma 1(a) and $f_\alpha \to f_\infty$ imply that (7.10) applies also to $\alpha = \infty$.) Since the functions f_α^* are convex, Lemma 4 will apply to $\bar{v}_\infty$ provided we can show $\bar{v}_\infty \in \operatorname{int}(\operatorname{dom} f_\infty^*)$. For this, observe that $k \leqslant f_\infty$ implies $f_\infty^* \leqslant k^*$, hence $\operatorname{int}(\operatorname{dom} k^*) \subset \operatorname{int}(\operatorname{dom} f_\infty^*)$. Furthermore, the assumed nonemptiness and boundedness of some lower level set of $k - \langle \cdot, \bar{v}_\infty \rangle$ is equivalent to $\bar{v}_\infty \in \operatorname{int}(\operatorname{dom} k^*)$. (See, for example, [40, Theorem 27.1(d)(f)], or the Moreau and Rockafellar results used in the proof of Theorem 8.) Therefore Lemma 4 implies there exist $M < +\infty$, $\mu > 0$, $\bar{\alpha}$ such that

$$f_\alpha^*(v) \leqslant M, \qquad \forall v \in \bar{v}_\infty + \mu B, \quad \forall \alpha = \bar{\alpha}, \dots, \infty. \tag{7.19}$$

It follows that

$$\bar{v}_\infty + \mu B \subset \operatorname{dom} f_\alpha^*, \qquad \forall \alpha = \bar{\alpha}, \dots, \infty, \tag{7.20}$$

$$h^* \leqslant f_\alpha^{**} \leqslant f_\alpha, \qquad \forall \alpha = \bar{\alpha}, \dots, \infty, \tag{7.21}$$

where h is the function defined on V by

$$h(v) = \begin{cases} M & \text{if } \; v \in \bar{v}_\infty + \mu B, \\ +\infty & \text{otherwise.} \end{cases}$$

By (7.21), Theorem 8 applies, yielding parts (a) and (b), and the Corollary to Theorem 8 applies also, yielding part (d). For (c), observe that from $f_\infty \not\equiv +\infty$ and Lemma 1(a) we have $\omega_\alpha(\cdot) < +\infty$ for all α sufficiently large. Then by (7.20) and (7.12) it follows that the ω_α, for all α sufficiently large, are finite everywhere on C and converge pointwise to ω_∞ there. Therefore, (c) follows from [40, Theorem 10.8]. Finally, consider part (e). Clearly, it suffices to show (7.17) just in the case of $0 < \varepsilon_\alpha \to 0$. Suppose it fails. Then there exist $v_\alpha \to v_\infty \in C$, $0 < \varepsilon_\alpha \to 0$, $\gamma > 0$ for which

$$\Omega_\beta(v_\beta, \varepsilon_\beta) \not\subset \Omega_\infty(v_\infty, 0) + \gamma B$$

occurs on some subsequence $(\beta) \subset (\alpha)$. For each $\beta \in (\beta)$ pick x_β such that

$$x_\beta \in \Omega_\beta(v_\beta, \varepsilon_\beta) \qquad \text{but} \qquad x_\beta \notin \Omega_\infty(v_\infty, 0) + \gamma B.$$

By (7.10), we can assume that $x_\beta \to x_\infty$ for some x_∞. Then (7.11) implies $x_\infty \in \Omega_\infty(v_\infty, 0)$. We have reached the absurdity

$$0 < \gamma < \|x_\beta - x_\infty\| \to 0.$$

Thus, (7.17) holds. For (7.18), let $v_\alpha \to v_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ be given. By (d),

$$(\omega_\infty)'_0(v_\infty; z) \leqslant \underline{\lim}(\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; z), \qquad \forall z \in V. \tag{7.22}$$

Furthermore, the functions $(\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; \cdot)$ for all α sufficiently large and for $\alpha = \infty$ are finite. Indeed (setting $\varepsilon_\infty = 0$ to cover $\alpha = \infty$ simultaneously), (5.16) yields

$$(\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; \cdot) = \inf\{\langle x, \cdot\rangle \mid x \in \bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha)\}. \tag{7.23}$$

Now pick $\tilde{\alpha} \geqslant \bar{\alpha}$ (where $\bar{\alpha}$ is as in (7.20)) so that $v_\alpha \in C$ and also $\omega_\alpha(\cdot) < +\infty$ for all $\alpha \geqslant \tilde{\alpha}$. Then for any $\alpha = \tilde{\alpha}, \ldots, \infty$ it follows from (7.20) that $v_\alpha \in \operatorname{int}(\operatorname{dom} f_\alpha^*)$, and hence from [40, Theorem 27.1(d)(f)] and (2.15) that $\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha)$ is nonempty and bounded. By (7.23), this shows $(\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; \cdot)$ is everywhere finite for all $\alpha = \tilde{\alpha}, \ldots, \infty$. Now let $\gamma > 0$ be given. By (7.22) and [40, Corollary 10.8.1] there exists $\hat{\alpha} \geqslant \tilde{\alpha}$ such that, for any $\alpha \geqslant \hat{\alpha}$,

$$(\omega_\infty)'_0(v_\infty; z) - \gamma \leqslant (\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; z), \qquad \forall z \in B.$$

By positive homogeneity (see (7.23)) this implies

$$(\omega_\infty)'_0(v_\infty; z) - \gamma \|z\| \leqslant (\omega_\alpha)'_{\varepsilon_\alpha}(v_\alpha; z), \qquad \forall z \in V,$$

and by (7.23) this yields

$$\psi^*_{\bar{\Omega}_\infty(v_\infty, 0) + \gamma B} = \psi^*_{\bar{\Omega}_\infty(v_\infty, 0)} + \gamma \psi^*_B \geqslant \psi^*_{\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha)}$$

(where ψ_S denotes the function having value 0 on S and $+\infty$ off S). Taking conjugates yields

$$\psi_{cl(\bar{\Omega}_\infty(v_\infty, 0) + \gamma B)} \leqslant \psi_{cl\,\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha)},$$

that is,

$$cl\,\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha) \subset cl(\bar{\Omega}_\infty(v_\infty, 0) + \gamma B).$$

Since the sets $\bar{\Omega}_\alpha(v_\alpha, \varepsilon_\alpha)$ are closed (as well as convex) and γB is compact, the closure operations here are redundant (e.g., [40, Corollary 9.1.2 and Theorem 8.4]). This establishes (7.18) and completes the proof.

Now consider the convex case.

THEOREM 13. *Assume that X is finite-dimensional and that $f_\alpha \to f_\infty$, where each function $f_1, f_2, \ldots, f_\infty$ is proper convex and lower semicontinuous. Let $\bar{v}_\infty$ be such that $\{x \in X \mid f_\infty(x) - \langle x, \bar{v}_\infty\rangle \leqslant \xi\}$ is nonempty and bounded for some $\xi \in R$. Then there exists $\mu > 0$ such that each of the following properties holds for $C = \{v \in V \mid \|v - \bar{v}_\infty\| < \mu\}$.*

(a) *Whenever* $v_\alpha \to v_\infty \in C$ *and* $0 \leqslant \varepsilon_\alpha \to 0$, *one has*

$$\varnothing \neq \overline{\lim}\, \Omega_\alpha(v_\alpha, \varepsilon_\alpha) \subset \Omega_\infty(v_\infty, 0), \tag{7.24}$$

$$\lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R. \tag{7.25}$$

(b) *More generally, whenever* (x_α), (v_α), v_∞, (ε_α) *satisfy*

$$x_\alpha \in \Omega_\alpha(v_\alpha, \varepsilon_\alpha), \qquad v_\alpha \to v_\infty \in C, \qquad 0 \leqslant \varepsilon_\alpha \to 0, \tag{7.26}$$

one has

$$\lim(f_\alpha(x_\alpha) - \langle x_\alpha, v_\alpha\rangle) = \lim \omega_\alpha(v_\alpha) = \omega_\infty(v_\infty) \in R \tag{7.27}$$

and the existence of $(\beta) \subset (\alpha)$ *and* x_∞ *such that*

$$x_\beta \to x_\infty \in \Omega_\infty(v_\infty, 0). \tag{7.28}$$

(c) *The functions* ω_α *converge pointwise to* ω_∞ *everywhere on* C, *and this convergence is uniform on* $\{v \in V \mid \|v - \bar{v}_\infty\| \leqslant \bar{\mu}\}$ *for every* $\bar{\mu} \in (0, \mu)$.

(d) *Whenever* $v_\alpha \to v_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ *and* $z_\alpha \to z_\infty$, *the approximate directional derivative functions* $\omega'_\varepsilon(v; z)$ *(defined in* (5.14)) *satisfy*

$$(\omega_\infty)'_0\,(v_\infty; z_\infty) \leqslant \underline{\lim}(\omega_\alpha)'_{\varepsilon_\alpha}\,(v_\alpha; z_\alpha). \tag{7.29}$$

(e) *Whenever* $v_\alpha \to v_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ *and* $\gamma > 0$, *there exists* $\hat{\alpha}$ *such that*

$$\Omega_\alpha(v_\alpha, \varepsilon_\alpha) \subset \Omega_\infty(v_\infty, 0) + \gamma B, \qquad \forall \alpha \geqslant \hat{\alpha}. \tag{7.30}$$

Proof. This is much the same as the proof of Theorem 12, so we only remark on a few points. The role of (7.10) and Lemma 3 is played here by convexity and Theorem 1. The refinement in part (a) from $0 < \varepsilon_\alpha \to 0$ to $0 \leqslant \varepsilon_\alpha \to 0$ follows from convexity of the f_α's. (See the end of the proof of Theorem 8 and also the proof of (6.10) in Proposition 3.) Lacking (7.10) here, the proof of (7.30) follows that of (7.18) rather than (7.17).

Some of the results in Theorem 13 can reportedly be derived using the recent results of Attouch and Wets [7], to which we have not had access.

We conclude by giving results which can be regarded as dual to those of Theorem 13.

THEOREM 14. *Assume that* X *is finite-dimensional and that* $f_\alpha \to f_\infty$, *where* f_α *is proper convex for all* $\alpha = \bar{\alpha}, \ldots, \infty$. *Let* $\bar{x}_\infty \in \operatorname{int}(\operatorname{dom} f_\infty)$. *Then there exists* $\mu > 0$ *such that each of the following properties holds for* $C = \{x \in X \mid \|x - \bar{x}_\infty\| < \mu\}$.

(a) *Whenever* $x_\alpha \to x_\infty \in C$ *and* $0 \leqslant \varepsilon_\alpha \to 0$, *one has*

$$\varnothing \neq \overline{\lim}\, \partial_{\varepsilon_\alpha} f_\alpha(x_\alpha) \subset \partial_0 f_\infty(x_\infty), \tag{7.31}$$

$$\lim f_\alpha(x_\alpha) = f_\infty(x_\infty) \in R. \tag{7.32}$$

(b) *More generally, whenever* (x_α), x_∞, (v_α), (ε_α) *satisfy*

$$v_\alpha \in \partial_{\varepsilon_\alpha} f_\alpha(x_\alpha), \qquad x_\alpha \to x_\infty \in C, \qquad 0 \leqslant \varepsilon_\alpha \to 0, \tag{7.33}$$

one has

$$\lim \inf_x \{f_\alpha(x) - \langle x - x_\alpha, v_\alpha\rangle\} = \lim f_\alpha(x_\alpha) = f_\infty(x_\infty) \in R \tag{7.34}$$

and the existence of $(\beta) \subset (\alpha)$ *and* v_∞ *such that*

$$v_\beta \to v_\infty \in \partial_0 f_\infty(x_\infty), \tag{7.35}$$

$$\lim(\langle x_\beta, v_\beta\rangle - f_\beta(x_\beta)) - f_\beta(x_\beta)) = \lim f_\beta^*(v_\beta) = f_\infty^*(v_\infty) \in R. \tag{7.36}$$

(c) *The functions* f_α *converge pointwise to* f_∞ *everywhere on* C, *and this convergence is uniform on* $\{x \in X \mid \|x - \bar{x}_\infty\| \leqslant \bar{\mu}\}$ *for every* $\bar{\mu} \in (0, \mu)$.

(d) *Whenever* $x_\alpha \to x_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ *and* $z_\alpha \to z_\infty$, *the approximate directional derivative functions* $f'_\varepsilon(x; z)$ (*defined in* (5.26)) *satisfy*

$$\overline{\lim}(f_\alpha)'_{\varepsilon_\alpha}(x_\alpha; z_\alpha) \leqslant (f_\infty)'_0(x_\infty; z_\infty). \tag{7.37}$$

(e) *Whenever* $x_\alpha \to x_\infty \in C$, $0 \leqslant \varepsilon_\alpha \to 0$ *and* $\gamma > 0$, *there exists* $\hat{\alpha}$ *such that*

$$\partial_{\varepsilon_\alpha} f_\alpha(x_\alpha) \subset \partial_0 f_\infty(x_\infty) + \gamma B, \qquad \forall \alpha \geqslant \hat{\alpha}. \tag{7.38}$$

Proof. Like that of Theorems 12 and 13. Theorem 9 and its Corollary play the role of Theorem 8 and its Corollary. Note that for part (e) one needs the fact that, for some $\tilde{\alpha}$, the sets $\partial_{\varepsilon_\alpha} f_\alpha(x_\alpha)$ are nonempty and bounded for all $\alpha = \tilde{\alpha}, \ldots, \infty$. For boundedness, see (5.25); for nonemptiness, observe that $\varnothing \neq \partial_0 f_\alpha(x_\alpha) \subset \partial_{\varepsilon_\alpha} f_\alpha(x_\alpha)$ follows from f_α being proper convex and bounded above around x_α (by (7.2)).

REFERENCES

1. E. ASPLUND, Fréchet differentiability of convex functions, *Acta Math.* **121** (1968), 31–47.
2. E. ASPLUND AND R. T. ROCKAFELLAR, Gradients of convex functions, *Trans. Amer. Math. Soc.* **139** (1969), 443–467.
3. H. ATTOUCH, "E.P.D. associées à des familles de sousdifférentiels," Thèse, Paris, 1976.

4. H. Attouch, Convergence de fonctionelles convexes, *in* "Journées d'analyse non linéaire" (P. Bénilan and J. Robert, Eds.), pp. 1–40, Springer-Verlag, Berlin/New York, 1978.
5. H. Attouch, Familles d'opérateurs maximaux monotones et measurabilité, *Ann. Mat. Pura Appl.* (4) **120** (1979), 35–111.
6. H. Attouch and C. Sbordone, Asymptotic limits for perturbed functionals of calculus of variations, *Ricerche Mat.* **29** (1980), 85–124.
7. H. Attouch and R. J.-B. Wets, Approximation and convergence in nonlinear optimization, *in* "Nonlinear Programming 4" (O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, Eds.), Academic Press, New York, 1981.
8. R. C. Bergstrom and L. McLinden, Convergent sequences of dual convex programs, in preparation.
9. L. Boccardo and P. Marcellini, Sulla convergenza delle soluzioni di disequazioni variazionali, *Ann. Mat. Pura Appl.* (4) **110** (1976), 137–159.
10. H. Brézis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert," Lecture Notes No. 5, North-Holland, Amsterdam, 1973.
11. A. Brøndsted, Conjugate convex functions in topological vector spaces, *Mat.-Fys. Medd. Danska Vid. Selsk.* **34** (1964), No. 2.
12. A. Brøndsted and R. T. Rockafellar, On the subdifferentiability of convex functions, *Proc. Amer. Math. Soc.* **16** (1965), 605–611.
13. L. Carbone and F. Colombini, On convergence of functionals with unilateral constraints, *J. Math. Pures Appl.* **59** (1980), 465–500.
14. M. M. Day, "Normed Linear Spaces," 3rd ed., Springer-Verlag, New York/Berlin, 1973.
15. E. De Giorgi, Convergence problems for functionals and operators, *in* "Recent Methods in Non-Linear Analysis" (E. De Giorgi, E. Magenes, and U. Mosco, Eds.), pp. 131–188, Pitagora Editrice, Bologna, 1979.
16. E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58** (1975), 842–850.
17. Z. Denkowski, The convergence of generalized sequences of sets and functions in locally convex spaces, I and II, *Zesgyty Nauk. Uniw. Jagiello. Prace Mat.* **22** (1981), 37–58 and 59–72.
18. Z. Denkowski, On the (Q)-convergence of nets of regular convex functions, *Zesgyty Nauk. Uniw. Jagiello. Prace Mat.* **22** (1981), 73–83.
19. P. Doktor, Perturbations of variational inequalities and rate of convergence of solutions, *Czechoslovak Math. J.* **30** (105) (1980), 426–437.
20. S. Dolecki, G. Salinetti, and R. J.-B. Wets, in preparation.
21. I. Ekeland and G. Lebourg, Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces, *Trans. Amer. Math. Soc.* **224** (1976), 193–216.
22. I. Ekeland and R. Temam, "Analyse convexe et problemes variationnels," Dunod, Paris, 1972.
23. W. Fenchel, On conjugate convex functions, *Canad. J. Math.* **1** (1949), 73–77.
24. J. L. Joly, "Une famille de topologies et de convergences sur l'ensemble des fonctionnelles convexes," Thèse, Université de Grenoble, 1970.
25. J. L. Joly, Une famille de topologies sur l'ensemble des fonctions convexes pour lesquelles la polarité est bicontinue, *J. Math. Pures Appl.* **52** (1973), 421–441.
26. P. Kosmol, Optimierung konvexer Funktionen mit Stabilitätsbetrachtungen, *Dissertationes Math.* (*Rozprawy Mat.*) **140** (1976), 5–38.
27. P. Kosmol, On stability of convex operators, *in* "Optimization and Operations Research" (R. Henn, B. Korte, and W. Oettli, Eds.), pp. 173–179, Springer-Verlag, Berlin/New York, 1978.
28. P.-J. Laurent, "Approximation et optimisation," Hermann, Paris, 1972.
29. P. Marcellini, Su una convergenza di funzioni convesse, *Boll. Un. Mat. Ital.* (4) **8** (1973), 137–158.

30. M. MATZEU, Su un tipo di continuità dell'operatore subdifferenziale, *Boll. Un. Mat. Ital. B* (5) **14** (1977), 480–490.
31. L. MCLINDEN AND R. C. BERGSTROM, Preservation of convergence of convex sets and functions in finite dimensions, *Trans. Amer. Math. Soc.* **268** (1981), 127–142.
32. J. J. MOREAU, Sur la fonction polaire d'une fonction semi-continue supérieurement, *C. R. Acad. Sci. Paris* **258** (1964), 1128–1131.
33. J. J. MOREAU, Fonctionelles convexes, Lecture Notes, Séminaire "Equations aux derivées partielles," Collège de France, Paris, 1966.
34. G. MOSCARIELLO, Γ-convergenza negli spazi sequenziali, *Rend. Accad. Sci. Fis. Mat. Napoli* (4) **43** (1976), 333–350.
35. U. MOSCO, Convergence of convex sets and of solutions of variational inequalies, *Advan. Math.* **3** (1969), 510–585.
36. U. MOSCO, On the continuity of the Young–Fenchel transform, *J. Math. Anal. Appl.* **35** (1971), 518–535.
37. R. ROBERT, Convergence de fonctionnelles convexes, *J. Math. Anal. Appl.* **45** (1974), 533–555.
38. R. T. ROCKAFELLAR, Level sets and continuity of conjugate convex functions, *Trans. Amer. Math. Soc.* **123** (1966), 46–63.
39. R. T. ROCKAFELLAR, Characterization of the subdifferentials of convex functions, *Pacific J. Math.* **17** (1966), 497–510.
40. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Mathematical Series, No. 28, Princeton Univ. Press, Princeton, N.J., 1970.
41. G. SALINETTI AND R. J.-B. WETS, On the relations between two types of convergence for convex functions, *J. Math. Anal. Appl.* **60** (1977), 211–226.
42. S. L. TROYANSKI, On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, *Studia Math.* **37** (1971), 173–180.
43. D. W. WALKUP AND R. J.-B. WETS, Continuity of some convex-cone-valued mappings, *Proc. Amer. Math. Soc.* **18** (1967), 229–235.
44. R. J.-B. WETS, Convergence of convex functions, variational inequalities, and convex optimization problems, *in* "Variational Inequalities and Complementarity Problems" (R. W. Cottle, F. Giannessi, and J.-L. Lions, Eds.), pp. 375–403, Wiley, Chichester, 1980.
45. R. A. WIJSMAN, Convergence of sequences of convex sets, cones and functions, *Bull. Amer. Math. Soc.* **70** (1964), 186–188.
46. R. A. WIJSMAN, Convergence of sequences of convex sets, cones and functions, II, *Trans. Amer. Math. Soc.* **123** (1966), 32–45.
47. K. YOSIDA, "Functional Analysis," 4th rev. ed., Springer-Verlag, New York/Berlin, 1974.
48. T. ZOLEZZI, On convergence of minima, *Boll. Un. Mat. Ital.* (4) **8** (1973), 246–257.
49. T. ZOLEZZI, "Approssimazioni e Perturbazioni di Problemi di Minimo," book in preparation.